Exercise 20 (Diagonal Systems). Consider the following system

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} x, \\
y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x,
\]

where \( c_1, c_2, \) and \( c_3 \) are unknown scalars.

(a) Provide an example of values for \( c_1, c_2, \) and \( c_3 \) for which the system is not observable.

(b) Provide an example of values for \( c_1, c_2, \) and \( c_3 \) for which the system is observable.

(c) Provide a necessary and sufficient condition on the \( c_i \) so that the system is observable.

Hint: Use the eigenvector test. Make sure that you provide a condition that when true the system is guaranteed to be observable, but when false the system is guaranteed not to be observable.

(d) Generalize the previous result for an arbitrary system with a single output and diagonal matrix \( A. \)

Solution to Exercise 20. (a) The system will not be observable, for example, for \( c_1 = c_2 = c_3 = 0, \) since in this case the observability matrix is zero.

(b) The system will be observable, for example, for \( c_1 = c_2 = c_3 = 1, \) since in this case the observability matrix is given by

\[
\mathcal{O} := \begin{bmatrix} 1 & 1 & 1 \\
1 & 0 & -1 \\
1 & 0 & 1 \end{bmatrix},
\]

which is nonsingular.

(c) The eigenvectors of \( A \) are given by

\[
\begin{bmatrix} \alpha \\
0 \\
0 \end{bmatrix}, \alpha \neq 0 \quad \begin{bmatrix} 0 \\
\beta \\
0 \end{bmatrix}, \beta \neq 0 \quad \begin{bmatrix} 0 \\
0 \\
\gamma \end{bmatrix}, \gamma \neq 0.
\]

For the system to be observable none of these can be in the kernel of the \( C \) so we need to have

\[
\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} = c_1 \neq 0, \quad \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\
1 \\
0 \end{bmatrix} = c_2 \neq 0, \quad \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix} = c_3 \neq 0.
\]

We thus conclude that a necessary and sufficient condition for the system to be observable is that

\( c_1 \neq 0 \) and \( c_2 \neq 0 \) \( \text{ and } c_3 \neq 0. \)

(d) When all the eigenvalues of a diagonal matrix \( A \in \mathbb{R}^{n \times n} \) are distinct (as in the prior example), all the eigenvectors of \( A \) are of the form

\[
\alpha e_i, \alpha \neq 0, \quad \forall i \in \{1, 2, \ldots, n\},
\]

where \( e_i \) denotes the \( i \)th elements of the canonical basis of \( \mathbb{R}^n \) (i.e., a vector with all entries equal to zero, except for the \( i \)th one, which is equal to 1). To have observability, none of
the eigenvectors can be in the kernel of \( c \). Therefore, a necessary and sufficient condition for observability is that none of the entries of the output vector \( c \) should be equal to zero.

Alternatively, if \( A \) is diagonal with (at least) one repeated eigenvalue, as in

\[
A = \begin{bmatrix}
\lambda & 0 & 0 & \cdots \\
0 & \lambda & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix},
\]

then \( A \) has eigenvalues of the form

\[
\alpha e_1 + \beta e_2, \; \alpha \neq 0 \text{ or } \beta \neq 0
\]

and for such vectors never to be in the kernel of \( c = [c_1 \; c_2 \; \cdots \; c_n] \), we need

\[
\alpha c_1 + \beta c_2 \neq 0, \quad (9)
\]

for all pairs \((\alpha, \beta)\) such that \( \alpha \neq 0 \) or \( \beta \neq 0 \). However, this is not possible because no matter what the values of \( c_1 \) and \( c_2 \), we can always find pairs \((\alpha, \beta)\) with \( \alpha \neq 0 \) or \( \beta \neq 0 \) that violate (9):

if \( c_1 = 0 \) pick \( \alpha = 1, \beta = 0 \)

if \( c_1 \neq 0 \) pick \( \beta = 1, \alpha = \frac{c_2}{c_1} \).

If the repeated eigenvalue were in the \( i \)th and \( j \)th diagonal entries of \( A \) (instead of the 1st and 2nd, as in the example above), we would have exactly the same problem with \( e_1 \) and \( e_2 \) replaced by \( e_i \) and \( e_j \).

In summary, a necessary and sufficient condition for observability is that the diagonal matrix \( A \) should not have repeated eigenvalues and all the entries of \( c \) should be nonzero.

**Exercise 21 (Diagonal Systems).** Consider the system

\[
\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x, \quad y = [c_1 \; c_2 \; c_3] x,
\]

where \( c_1, c_2, \) and \( c_3 \) are unknown scalars.

(a) Provide a necessary and sufficient condition on the \( c_i \) so that the system is detectable.

*Hint: Use the eigenvector test. Make sure that you provide a condition that when true the system is guaranteed to be detectable, but when false the system is guaranteed not to be detectable.*

(b) Generalize the previous result for an arbitrary system with a single output and diagonal matrix \( A \).

---

**Solution to Exercise 21.** (a) The eigenvectors of \( A \) corresponding to the eigenvalues with positive or zero real part are given by

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

For the system to be detectable none of these can be in the kernel of the \( C \) so we need to have

\[
\begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = c_1 \neq 0, \quad \begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = c_2 \neq 0.
\]

We thus conclude that a necessary and sufficient condition for the system to be detectable is that

\( c_1 \neq 0, \; c_2 \neq 0. \)
(b) For detectability, we only need to concern ourselves with eigenvectors that correspond to eigenvalues that are positive or zero. When all the positive or zero eigenvalues of a diagonal matrix $A \in \mathbb{R}^{n \times n}$ are distinct (like in the example above), all the corresponding eigenvectors of $A$ are of the form

$$\alpha e_i, \alpha \neq 0,$$

where $e_i$ denotes the $i$th elements of the canonical basis of $\mathbb{R}^n$ (i.e., a vector with all entries equal to zero, except for the $i$th one, which is equal to 1). To have detectability, none of these eigenvectors can be in the kernel of $c$. Therefore, a necessary and sufficient condition for detectability is that none of the entries of the output vector $c$ corresponding to positive or zero diagonal entries of $A$ should be equal to zero.

Alternatively, if $A$ is diagonal with a zero or positive repeated eigenvalue, as in

$$A = \begin{bmatrix} \lambda & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \lambda \geq 0$$

then $A$ has eigenvalues of the form

$$\alpha e_1 + \beta e_2, \alpha \neq 0 \text{ or } \beta \neq 0$$

and for such vectors never to be in the kernel of $c = [c_1 \ c_2 \ \cdots \ c_n]$, we need

$$\alpha c_1 + \beta c_2 \neq 0,$$

for all pairs $(\alpha, \beta)$ such that $\alpha \neq 0$ or $\beta \neq 0$. However, this is not possible because no matter what are the values of $c_1$ and $c_2$, we can always find pairs $(\alpha, \beta)$ with $\alpha \neq 0$ or $\beta \neq 0$ that violate (10):

- if $c_1 = 0$ pick $\alpha = 1, \beta = 0$
- if $c_1 \neq 0$ pick $\beta = 1, \alpha = -\frac{c_2}{c_1}$

In summary, the necessary and sufficient condition for detectability is that the diagonal matrix $A$ should not have repeated eigenvalues that are zero or positive and all the entries of $c$ corresponding to positive or zero diagonal entries of $A$ should be nonzero. ■

Exercise 22 (Repeated eigenvalues). Consider the SISO LTI system

$$\dot{x} + Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, u, y \in \mathbb{R}.$$

(a) Assume that $A$ is a diagonal matrix and $B$, $C$ are column/row vectors with entries $b_i$ and $c_j$, respectively. Write the controllability and observability matrices for this system.

(b) Show that if $A$ is a diagonal matrix with repeated eigenvalues, then the pair $(A, B)$ cannot be controllable and the pair $(A, C)$ cannot be observable.

(c) Given a SISO transfer function $T(s)$, can you find a minimal realization for $T(s)$ for which the matrix $A$ is diagonalizable with repeated eigenvalues? Justify your answer.

(d) Given a SISO transfer function $T(s)$, can you find a minimal realization for $T(s)$ for which the matrix $A$ is not diagonalizable with repeated eigenvalues? Justify your answer.

Hint: An example suffices to justify the answer “yes” in (c) or (d).
Solution to Exercise 22. (a) Since

\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots \\
0 & \lambda_2 & 0 & \cdots \\
0 & 0 & \lambda_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots
\end{bmatrix}_{n \times k}, \quad C = \begin{bmatrix}
c_1 & c_2 & c_3 & \cdots
\end{bmatrix}_{m \times n},
\]

the controllability and observability matrices are given by

\[
\mathcal{C} := \begin{bmatrix}
B & AB & A^2B & \cdots
\end{bmatrix} = \begin{bmatrix}
b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots \\
b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \cdots \\
b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
\mathcal{O} := \begin{bmatrix}
C \\
CA \\
CA^2
\end{bmatrix} = \begin{bmatrix}
c_1 & c_2 & c_3 & \cdots \\
\lambda_1 c_1 & \lambda_2 c_2 & \lambda_3 c_3 & \cdots \\
\lambda_1^2 c_1 & \lambda_2^2 c_2 & \lambda_3^2 c_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

(b) Assuming that \( \lambda_1 = \lambda_2 = \lambda \) the first two rows of \( \mathcal{C} \) are linearly dependent since they are both scaled versions of

\[
\begin{bmatrix}
1 \\
\lambda \\
\lambda^2 \\
\vdots
\end{bmatrix}_{1 \times n-1}.
\]

Therefore \( \mathcal{C} \) cannot have rank \( n \) and the system cannot be controllable. Similarly, the first two columns of \( \mathcal{O} \) are linearly dependent and therefore the system cannot be observable.

(c) No, because if this was possible we could write down the system in the coordinates for which \( A \) is diagonal and, from the exercise above, the system would not be controllable nor observable. Note that controllability and observability are invariant with respect to equivalence transformations.

(d) Yes, for example

\[
A = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad c = \begin{bmatrix}
1 & 0
\end{bmatrix}.
\]

The controllability and observability matrices for this system are

\[
\mathcal{C} = \begin{bmatrix}
b \\
Ab
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix}
c \\
ca
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Since both matrices are nonsingular the system is both controllable and observable and therefore the realization is minimal.

\[\blacksquare\]
Homework #7

Student name:

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<tr>
<th>Exercise</th>
<th>weight</th>
<th>grade (0..4)</th>
</tr>
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<tr>
<td>20 (b)</td>
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</tbody>
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Grades

4 – correct
3 – mostly correct
2 – half-and-half
1 – mostly incorrect
0 – not done

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\[1\] You can give yourself full credit if you simply forgot to consider the case of repeated eigenvalues.