LINEAR SYSTEMS THEORY

2ND EDITION

João P. Hespanha

December 22, 2017

Disclaimer: This is a draft and probably contains a few typos.
Comments and information about typos are welcome.
Please contact the author at hespanha@ece.ucsb.edu.

© Copyright to João Hespanha. Please do not distribute this document without the author’s consent.
To my wife Stacy
and
to our son Rui
# Contents

Preamble vii

Linear Systems I — Basic Concepts 3

I System Representation 3

1 State-Space Linear Systems 5
   1.1 State-Space Linear Systems ........................................ 5
   1.2 Block Diagrams .................................................. 7
   1.3 Exercises ....................................................... 10

2 Linearization 11
   2.1 State-Space Nonlinear Systems .................................... 11
   2.2 Local Linearization Around an Equilibrium Point ............. 11
   2.3 Local Linearization Around a Trajectory ...................... 13
   2.4 Feedback Linearization .......................................... 14
   2.5 Practice Exercises ............................................. 18
   2.6 Exercises ....................................................... 22

3 Causality, Time Invariance, and Linearity 25
   3.1 Basic Properties of LTV/LTI Systems .......................... 25
   3.2 Characterization of All Outputs to a Given Input ............ 27
   3.3 Impulse Response ................................................ 27
   3.4 Laplace and $\mathcal{Z}$ Transforms (review) ................... 30
   3.5 Transfer Function ................................................ 30
   3.6 Discrete-Time Case .............................................. 31
   3.7 Additional Notes ................................................ 32
   3.8 Exercises ....................................................... 33

4 Impulse Response and Transfer Function of State-Space Systems 35
   4.1 Impulse Response and Transfer Function for LTI Systems .... 35
   4.2 Discrete-Time Case ............................................. 36
   4.3 Elementary Realization Theory .................................. 36
   4.4 Equivalent State-Space Systems ................................ 40
   4.5 LTI Systems in MATLAB® ........................................ 40
   4.6 Practice Exercises ............................................. 42
   4.7 Exercises ....................................................... 43

5 Solutions to LTV Systems 45
   5.1 Solution to Homogeneous Linear Systems ...................... 45
   5.2 Solution to Nonhomogeneous Linear Systems .................. 47
   5.3 Discrete-Time Case ............................................. 48
III  Controllability and State Feedback 105

11  Controllable and Reachable Subspaces 107
   11.1 Controllable and Reachable Subspaces .......................... 107
   11.2 Physical Examples and System Interconnections ................. 108
   11.3 Fundamental Theorem of Linear Equations (Review) ........... 111
   11.4 Reachability and Controllability Gramians ...................... 111
   11.5 Open-Loop Minimum-Energy Control .............................. 113
   11.6 Controllability Matrix (LTI) .................................... 114
   11.7 Discrete-Time Case ............................................. 116
   11.8 MATLAB® Commands ............................................. 119
   11.9 Practice Exercise .............................................. 119
   11.10 Exercises ..................................................... 121

12  Controllable Systems 123
   12.1 Controllable Systems ............................................. 123
   12.2 Eigenvector Test for Controllability ............................ 124
   12.3 Lyapunov Test for Controllability ................................ 126
   12.4 Feedback Stabilization Based on the Lyapunov Test ............. 128
   12.5 Eigenvalue Assignment .......................................... 129
   12.6 Practice Exercises .............................................. 130
   12.7 Exercises ..................................................... 131

13  Controllable Decompositions 135
   13.1 Invariance with Respect to Similarity Transformations ........ 135
   13.2 Controllable Decomposition ..................................... 135
   13.3 Block Diagram Interpretation ................................... 137
   13.4 Transfer Function ............................................... 138
   13.5 MATLAB® Commands ............................................. 138
   13.6 Exercise ....................................................... 138

14  Stabilizability 141
   14.1 Stabilizable System ............................................. 141
   14.2 Eigenvector Test for Stabilizability ............................ 142
   14.3 Popov-Belevitch-Hautus (PBH) Test for Stabilizability ......... 143
   14.4 Lyapunov Test for Stabilizability ................................ 144
   14.5 Feedback Stabilization Based on the Lyapunov Test ............. 145
   14.6 MATLAB® Commands ............................................. 145
   14.7 Exercises ..................................................... 146

IV  Observability and Output Feedback 147

15  Observability 149
   15.1 Motivation: Output Feedback .................................... 149
   15.2 Unobservable Subspace .......................................... 150
   15.3 Unconstructible Subspace ....................................... 151
   15.4 Physical Examples ............................................... 151
   15.5 Observability and Constructibility Gramians ..................... 153
   15.6 Gramian-Based Reconstruction ................................... 154
   15.7 Discrete-Time Case ............................................. 155
   15.8 Duality for LTI Systems ......................................... 156
   15.9 Observability Tests ............................................. 157
   15.10 MATLAB® Commands ............................................. 158
   15.11 Practice Exercises .............................................. 160
15.12 Exercises .......................................................... 162

16 Output Feedback ....................................................... 165
   16.1 Observable Decomposition ....................................... 165
   16.2 Kalman Decomposition Theorem ................................. 166
   16.3 Detectability ....................................................... 168
   16.4 Detectability Tests ................................................ 170
   16.5 State Estimation ................................................... 170
   16.6 Eigenvalue Assignment by Output Injection ...................... 171
   16.7 Stabilization through Output Feedback ............................. 172
   16.8 MATLAB® Commands .............................................. 173
   16.9 Exercises ........................................................... 173

17 Minimal Realizations .................................................. 175
   17.1 Minimal Realizations ............................................... 175
   17.2 Markov Parameters .................................................. 176
   17.3 Similarity of Minimal Realizations ................................. 177
   17.4 Order of a Minimal SISO Realization ............................... 179
   17.5 MATLAB® Commands .............................................. 180
   17.6 Practice Exercises .................................................. 180
   17.7 Exercises ........................................................... 182

Linear Systems II — Advanced Material .................................. 185

V Poles and Zeros of MIMO Systems ...................................... 185

18 Smith-McMillan Form .................................................. 187
   18.1 Informal Definition of Poles and Zeros ............................ 187
   18.2 Polynomial Matrices: Smith Form .................................. 188
   18.3 Rational Matrices: Smith-McMillan Form ........................... 190
   18.4 McMillan Degree, Poles, and Zeros ............................... 191
   18.5 Blocking Property of Transmission Zeros ........................... 192
   18.6 MATLAB® Commands .............................................. 193
   18.7 Exercises ........................................................... 193

19 State-Space Poles, Zeros, and Minimality ............................. 195
   19.1 Poles of Transfer Functions versus Eigenvalues of State-Space Realizations . . . 195
   19.2 Transmission Zeros of Transfer Functions versus Invariant Zeros of State-Space Realizations .................................................. 196
   19.3 Order of Minimal Realizations ....................................... 198
   19.4 Practice Exercises .................................................. 199
   19.5 Exercise ............................................................. 200

20 System Inverses ........................................................ 203
   20.1 System Inverse ....................................................... 203
   20.2 Existence of an Inverse .............................................. 204
   20.3 Poles and Zeros of an Inverse ....................................... 205
   20.4 Feedback Control of Invertible Stable Systems with Stable Inverses ............... 206
   20.5 MATLAB® Commands .............................................. 208
   20.6 Exercises ........................................................... 208
# VI LQR/LQG Optimal Control

## 21 Linear Quadratic Regulation (LQR)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.1 Deterministic Linear Quadratic Regulation (LQR)</td>
<td>211</td>
</tr>
<tr>
<td>21.2 Optimal Regulation</td>
<td>212</td>
</tr>
<tr>
<td>21.3 Feedback Invariants</td>
<td>213</td>
</tr>
<tr>
<td>21.4 Feedback Invariants in Optimal Control</td>
<td>213</td>
</tr>
<tr>
<td>21.5 Optimal State Feedback</td>
<td>213</td>
</tr>
<tr>
<td>21.6 LQR in MATLAB®</td>
<td>215</td>
</tr>
<tr>
<td>21.7 Additional Notes</td>
<td>215</td>
</tr>
<tr>
<td>21.8 Exercises</td>
<td>216</td>
</tr>
</tbody>
</table>

## 22 The Algebraic Riccati Equation (ARE)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.1 The Hamiltonian Matrix</td>
<td>217</td>
</tr>
<tr>
<td>22.2 Domain of the Riccati Operator</td>
<td>217</td>
</tr>
<tr>
<td>22.3 Stable Subspaces</td>
<td>218</td>
</tr>
<tr>
<td>22.4 Stable Subspace of the Hamiltonian Matrix</td>
<td>219</td>
</tr>
<tr>
<td>22.5 Exercises</td>
<td>222</td>
</tr>
</tbody>
</table>

## 23 Frequency Domain and Asymptotic Properties of LQR

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.1 Kalman’s Equality</td>
<td>223</td>
</tr>
<tr>
<td>23.2 Frequency Domain Properties: Single-Input Case</td>
<td>224</td>
</tr>
<tr>
<td>23.3 Loop Shaping Using LQR: Single-Input Case</td>
<td>226</td>
</tr>
<tr>
<td>23.4 LQR Design Example</td>
<td>228</td>
</tr>
<tr>
<td>23.5 Cheap Control Case</td>
<td>230</td>
</tr>
<tr>
<td>23.6 MATLAB® Commands</td>
<td>233</td>
</tr>
<tr>
<td>23.7 Additional Notes</td>
<td>233</td>
</tr>
<tr>
<td>23.8 The Loop-Shaping Design Method (Review)</td>
<td>234</td>
</tr>
<tr>
<td>23.9 Exercises</td>
<td>237</td>
</tr>
</tbody>
</table>

## 24 Output Feedback

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.1 Certainty Equivalence</td>
<td>239</td>
</tr>
<tr>
<td>24.2 Deterministic Minimum-Energy Estimation (MEE)</td>
<td>240</td>
</tr>
<tr>
<td>24.3 Stochastic Linear Quadratic Gaussian (LQG) Estimation</td>
<td>244</td>
</tr>
<tr>
<td>24.4 LQR/LQG Output Feedback</td>
<td>244</td>
</tr>
<tr>
<td>24.5 Loop Transfer Recovery (LTR)</td>
<td>245</td>
</tr>
<tr>
<td>24.6 Optimal Set-Point Control</td>
<td>246</td>
</tr>
<tr>
<td>24.7 LQR/LQG with MATLAB®</td>
<td>249</td>
</tr>
<tr>
<td>24.8 LTR Design Example</td>
<td>250</td>
</tr>
<tr>
<td>24.9 Exercises</td>
<td>251</td>
</tr>
</tbody>
</table>

## 25 LQG/LQR and the \( Q \) Parameterization

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.1 ( Q )-Augmented LQG/LQR Controller</td>
<td>253</td>
</tr>
<tr>
<td>25.2 Properties</td>
<td>254</td>
</tr>
<tr>
<td>25.3 ( Q ) Parameterization</td>
<td>256</td>
</tr>
<tr>
<td>25.4 Exercise</td>
<td>257</td>
</tr>
</tbody>
</table>

## 26 \( Q \) Design

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>26.1 Control Specifications for ( Q ) Design</td>
<td>259</td>
</tr>
<tr>
<td>26.2 The ( Q ) Design Feasibility Problem</td>
<td>262</td>
</tr>
<tr>
<td>26.3 Finite-Dimensional Optimization: Ritz Approximation</td>
<td>262</td>
</tr>
<tr>
<td>26.4 ( Q ) Design Using MATLAB® and CVX</td>
<td>263</td>
</tr>
<tr>
<td>26.5 ( Q ) Design Example</td>
<td>268</td>
</tr>
<tr>
<td>26.6 Exercise</td>
<td>269</td>
</tr>
</tbody>
</table>
Preamble

*Linear systems theory* is the cornerstone of control theory and a prerequisite for essentially all graduate courses in this area. It is a well-established discipline that focuses on linear differential equations from the perspective of control and estimation.

Content

The first set of lectures (1–17) covers the key topics in linear systems theory: system representation, stability, controllability and state feedback, observability and state estimation, and realization theory. The main goal of these chapters is to provide the background needed for advanced control design techniques. Feedback linearization and the LQR problem are also briefly introduced to increase the design component of this set of lectures. The preview of optimal LQR control facilitates the introduction of notions such as controllability and observability, but is pursued in much greater detail in the second set of lectures.

Three advanced foundational topics are covered in a second set of lectures (18–26): poles and zeros for MIMO systems, LQG/LQR control, and control design based on the $Q$ parameterization of stabilizing controllers ($Q$ design). The main goal of these chapters is to introduce advanced supporting material for modern control design techniques. Although LQG/LQR is covered in some other linear systems books, it is generally not covered at the same level of detail (in particular the frequency domain properties of LQG/LQR, loop shaping, and loop transfer recovery). In fact, there are few textbooks in print that cover the same material, in spite of the fact that these are classical results and LQG/LQR is the most widely used form of state-space control. By covering the ARE in detail, I set the stage for H-2 and H-infinity.

In writing this book, it is assumed that the reader is familiar with linear algebra and ordinary differential equations at an undergraduate level. To profit most from this textbook, the reader would also have taken an undergraduate course in classical control, but these notes are basically self-contained regarding control concepts.

Organization and Style

This book was purposely designed as a textbook, and because it is not an adaptation of a reference text, the main emphasis is on presenting material in a fashion that makes it easy for students to understand. The material is organized in lectures, and it is divided so that on average each lecture can be covered in 2 hours of class time. The sequence in which the material appears was selected to emphasize continuity and motivate the need for new concepts as they are introduced.

In writing this manuscript there was a conscious effort to reduce verbosity. This is not to say that I did not attempt to motivate the concepts or discuss their significance (on the contrary), but the amount of text was kept to a minimum. Typically, discussion, remarks, and side comments are relegated to marginal notes so that the reader can easily follow the material presented without distraction and yet enjoy the benefit of comments on the notation and terminology, or be made aware that a there is a related MATLAB® command.

Attention! When a marginal note finishes with “p. XXX,” more information about that topic can be found on page XXX.
I have also not included a chapter or appendix that summarizes background material (for example, a section on linear algebra or nonlinear differential equations). Linear algebra is a key prerequisite to this course, and it is my experience that referring a student who has a limited background in linear algebra to a brief chapter on the subject is useless (and sometime even counter-productive). I do review advanced concepts (for example, singular values, matrix norms, and the Jordan normal form), but this is done at the points in the text where these concepts are needed. I also take this approach when referring the reader to MATLAB®, by introducing the commands only where the relevant concepts appear in the text.

Learning and Teaching Using This Textbook

Lectures 1–17 can be the basis for a one-quarter graduate course on linear systems theory. At the University of California at Santa Barbara I teach essentially all the material in these lectures in one quarter with about 40 hours of class time. In the interest of time, the material in the Additional Notes sections and some of the discrete-time proofs can be skipped. For a semester-long course, one could also include a selection of the advanced topics covered in the second part of the book (Lectures 18–26).

I have tailored the organization of the textbook to simplify the teaching and learning of the material. In particular, the sequence of the chapters emphasizes continuity, with each chapter appearing motivated and in logical sequence with the preceding ones. I always avoid introducing a concept in one chapter and using it again only many chapters later. It has been my experience that even if this may be economical in terms of space, it is pedagogically counterproductive. The chapters are balanced in length so that on average each can be covered in roughly 2 hours of lecture time. Not only does this aid the instructor’s planning, but it makes it easier for the students to review the materials taught in class.

As I have taught this material, I have noticed that some students start graduate school without proper training in formal reasoning. In particular, many students come with limited understanding of the basic logical arguments behind mathematical proofs. A course in linear systems provides a superb opportunity to overcome this difficulty. To this effect, I have annotated several proofs with marginal notes that explain general techniques for constructing proofs: contradiction, contraposition, the difference between necessity and sufficiency, etc. (see, for example, Note 14 on page 91). Throughout the manuscript, I have also structured the proofs to make them as intuitive as possible, rather than simply as short as possible. All mathematical derivations emphasize the aspects that give insight into the material presented and do not dwell on technical aspects of small consequence that merely bore the students. Often these technical details are relegated to marginal notes or exercises.

The book includes exercises that should be solved as the reader progresses through the material. Some of these exercises clarify issues raised in the body of the text and the reader is generally pointed to such exercises in marginal notes; for example, Exercise 8.5, which is referenced in a marginal note in page 76. Other exercises are aimed at consolidating the knowledge acquired, by asking the reader to apply algorithms or approaches previously discussed; for example, Exercise 2.10 on page 24. The book includes detailed solutions for all the exercises that appear in the sections titled “Practice Exercises,” but it does not include solutions to those in the sections titled “Additional Exercises,” which may be used for assignments or quizzes.

MATLAB®

Computational tools such as the MATLAB® software environment offer a significant step forward in teaching linear systems because they allow students to solve numerical problems without being exposed to a detailed treatment of numerical computations. By systematically annotating the theoretical developments with marginal notes that discuss the relevant commands available in MATLAB®, this textbook helps students learn to use these tools. An example of this can be found in MATLAB® Hint 9 on page 12, which is further expanded on page 57.
The commands discussed in the “MATLAB® Hints” assume that the reader has version R2015b of MATLAB® with Simulink®, the Symbolic Math Toolbox, and the Control System Toolbox. However, essentially all these commands have been fairly stable for several versions so they are likely to work with previous and subsequent versions for several years to come. Lecture 26 assumes that the reader has installed CVX version 1.2, which is a MATLAB® package for Disciplined Convex Programming, distributed under the GNU General Public License 2.0 [7].

MATLAB® and Simulink® are registered trademarks of The MathWorks Inc. and are used with permission. The MathWorks does not warrant the accuracy of the text or exercises in this book. This book’s use or discussion of MATLAB®, Simulink®, or related products does not constitute an endorsement or sponsorship by The MathWorks of a particular pedagogical approach or particular use of the MATLAB® and Simulink® software.

Web

The reader is referred to the author’s website at www.ece.ucsb.edu/~hespanha for corrections, updates on MATLAB® and CVX, and other supplemental material.

2nd Edition

The main thrust for publishing a second edition of this book was to facilitate the process of learning this material, both in the classroom or in a self-teaching environment. This second edition covers the same basic theoretical concepts, but we have added a large number of exercises, including solved exercises that appear in the sections titled “Practice Exercises.” These exercises are based on prototypical problems and guide the student toward answers that are correct, precisely stated, and succinct. We also took this opportunity to reorganize a few lectures and correct many typos. Most typos were annoying but inconsequential, but a few could be misleading. A special thanks to all the students that got back to me with typos, requests for clarification, and other suggestions for improvement.

Acknowledgments

Several friends and colleagues have helped me improve this manuscript through their thoughtful constructive comments and suggestions. Among these, I owe special thanks to A. Pedro Aguiar, Karl Åström, Stephen Boyd, Bassam Bamieh, Maurice Heemels, Mustafa Khammash, Daniel Klein, Petar Kokotović, Kristi Morgansen, Dale Seborg, and Emre Tuna, as well as all the students who used early drafts of these notes and provided me with numerous comments and suggestions. I would also like to acknowledge the support of several organizations, including the National Science Foundation (NSF), the Army Research Office (ARO), the Air Force Office of Scientific Research (AFOSR), and the University of California at Santa Barbara.