STABILIZATION THROUGH HYBRID CONTROL

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Contents

Summary

This chapter addresses the problem of controlling a dynamical process using a *hybrid controller*, i.e., a controller that combines continuous dynamics with discrete logic. Typically, the discrete logic is used to effectively switch between several continuous controls laws and is called a *supervisor*. We review several tools that can be found in the literature to design this type of hybrid controllers and to analyze the resulting closed-loop system. We illustrate how these tools can be utilized through two case studies.

1. Introduction

The basic problem considered here is the control of complex systems for which traditional control methodologies based on a single continuous controller do not provide satisfactory performance. In *hybrid control*, one builds a bank of alternative candidate controllers and switches among them based on measurements collected online. The switching is orchestrated by a specially designed logic that uses the measurements to decide which controller should be placed in the feedback loop at each instant of time. Figure 1 shows the basic architecture employed by hybrid control.

In this figure u represents the control input, d an exogenous disturbance and/or measurement noise, and y the measured output. The dashed box is a conceptual representation of a switching controller. In practice, switching controllers are implemented differently. Suppose that we desire to switch among a family $\mathcal{C}$ of controllers parameterized by some variable $q \in \mathcal{Q}$. For example, we could have

$$\mathcal{C} := \big\{ \dot{z}_q = F_q(z_q, y),\ u = G_q(z_q, y) : q \in \mathcal{Q} \big\},$$

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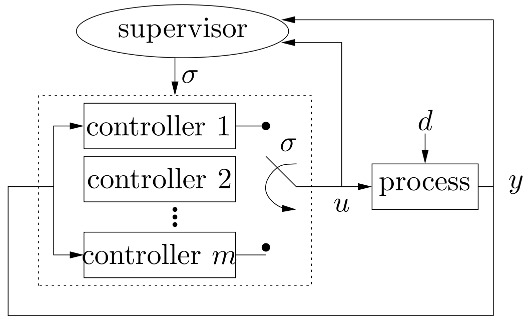

Figure 1: Hybrid control

where the set $\mathcal{Q}$ that parameterizes the functions $F_q(\cdot)$, $G_q(\cdot)$, $q \in \mathcal{Q}$ can be finite, infinite but countable, or not even countable (e.g., a ball in $\mathbb{R}^k$). Switching among the controllers in $\mathcal{C}$ can then be accomplished using the following *multi-controller*:

$$\dot{x}_C = F_\sigma(x_C, y), \qquad\qquad u = G_\sigma(x_C, y), \tag{1}$$

where $\sigma : [0, \infty) \to \mathcal{Q}$ is a piecewise constant signal—called the *switching signal*—that effectively determines which controller is in the loop at each instant of time. The points of discontinuity of σ correspond to a change in candidate controller and are therefore called *switching times*. The multi-controller in (1) is far more efficient than the conceptual structure in Figure 1 as its dimension is independent of the number of candidate controllers. Moreover, if some of the controllers in Figure 1 were unstable, their interval states could become unbounded if they were left out of the feedback loop. These issues are further discussed by Morse (1995). In this chapter, we use a continuous-time multi-controller such as (1) to keep the exposition concrete. However, the concepts presented generalize to other types of candidate control laws, such as discrete-time (Borrelli *et al.*, 1998) or hybrid controllers (Hespanha *et al.*, 1999).

The top element in Figure 1 is the logic that controls the switch, or more precisely, that generates the switching signal in (1). This logic is called the *supervisor* and its purpose is to monitor the signals that can be measured (in this case u and y) and decide, at each instant of time, which candidate controller should be put in the feedback loop with the process. In hybrid control, the supervisor combines continuous dynamics with discrete logic and is therefore a *hybrid system*. A typical hybrid supervisor can be defined by a an ordinary differential equation coupled with a recursive equation such as

$$\dot{\varphi} = \Psi_\sigma(\varphi, u, y), \qquad\qquad \sigma = \Gamma(\varphi, \sigma^-), \tag{2}$$

where $\{\Psi_q(\cdot) : q \in \mathcal{Q}\}$ is a family of vector fields, and $\Gamma(\cdot)$ a discrete transition function. A pair of signals (φ, σ) is called a *solution* to (2) if σ is piecewise constant taking values in $\mathcal{Q}$, φ is a solution in the sense of Carathéodory to the time-varying differential equation

$$\dot{\varphi} = \Psi_{\sigma(t)}\big(\varphi, u(t), y(t)\big), \quad t > 0$$

and, for every $t > 0$,

$$\sigma(t) = \Gamma\big(\varphi(t), \sigma^-(t)\big).$$

The signal φ is called the *continuous state* of the supervisor and σ its *discrete state*. We assume here that all signals of interest are continuous from above, and, given a piecewise continuous signal σ, we denote by σ^- the signal defined by $\sigma^-(t) = \lim_{\tau \uparrow t} \sigma(\tau)$, $t > 0$. More general models for hybrid systems and more sophisticated notions of solution can be found in Chapter

Modeling of Hybrid Systems and in the work of Tavernini (1987); Morse *et al.* (1992); Back *et al.* (1993); Nerode and Kohn (1993); Antsaklis *et al.* (1993); Brockett (1993); Branicky *et al.* (1994); Lygeros *et al.* (1999); Zhang *et al.* (2000).

Hybrid control systems, like the one depicted in Figure 1, are used in many situations, such as:

1. When the performance requirements for the closed-loop system change over time. In this case, the supervisor is responsible for placing in the feedback loop the controller that is most suitable for the current needs.

2. When there is large uncertainty in the process to be controlled and offline identification is not possible or desirable. Here, the supervisor should place in the feedback loop the controller that is more likely to stabilize the actual process and provide adequate performance. This type of hybrid control can be viewed as a form of adaptive control, where switching replaces the more traditional continuous tuning. This type of hybrid control is considered in the case study in Section 4.2.

3. When the nature of the process requires hybrid stabilization. This can occur because there are fundamental limitations on the type of controllers that are able to stabilize the process or because the actuation or sensing mechanisms naturally result in switching control laws. Examples of the former are nonholonomic systems (cf., *Control of Nonlinear Systems* and Brockett, 1983) and of the later are systems for which actuation is achieved through on-off valves or switches, or when the sensors used for feedback have a limited range of operation (cf. case study in Section 4.1).

The reader is referred to the work of Morse (1995); Hespanha (1998); Eker and Malmborg (1999); Lemmon *et al.* (1999); Liberzon and Morse (1999); DeCarlo *et al.* (2000) and references therein for additional examples.

The interconnection of a process modeled by an ordinary differential equation, the multi-controller (1), and the hybrid supervisor (2), results in a hybrid system of the form

$$\dot{x} = A_\sigma(x, d), \qquad\qquad \sigma = \Phi(x, \sigma^-), \tag{3}$$

where the continuous state x takes value in $\mathbb{R}^n$, the discrete state σ is the switching signal that takes values in $\mathcal{Q}$, and d the process' exogenous disturbance. The analysis of this type of systems has been actively pursued in the last years. In particular, considerable research has been carried out to answer: reachability questions such has

> *Given two disjoint sets $\mathcal{S}, \mathcal{R} \subset \mathbb{R}^n \times \mathcal{Q}$, if the state (x, σ) of* (3) *starts inside $\mathcal{S}$, will it ever enter $\mathcal{R}$?*

liveness questions such has

> *Given two discrete states $q_1, q_2 \in \mathcal{Q}$, will there be an infinite number of switching times at which σ switches from q_1 to q_2?*

or stability questions such as

> *Will the solution to* (3) *exist globally and, if so, will the continuous state x remain uniformly bounded and the output y converge to some set-point r as $t \to \infty$?*

In this chapter we are mostly interested in stability questions such as the last one. Note that with hybrid systems like (3), global existence of solution may fail either because the continuous state x becomes unbounded in finite time—often called *finite escape time*—or because the discrete state σ exhibits an infinite number of switches in finite time—often called *chattering* or the *Zeno phenomenon* (cf. *Modeling of Hybrid Systems*, *Well-posedness of Hybrid Systems*, and Johansson *et al.*, 1999).

There is no systematic procedure to study the stability of a generic hybrid system. However, the arguments used to prove the stability of hybrid systems usually consist of consecutively applying results of the type

PD: *Assuming that x belongs to a family $\mathcal{X}_k$ of signals taking values in $\mathbb{R}^n$, then the discrete state σ belong to the family $\mathcal{S}_k$ of switching signals.*

PC: *Assuming that σ belongs to a family $\mathcal{S}_k$ of switching signals, then the continuous state x belongs to the family $\mathcal{X}_{k+1}$ of signals taking values in $\mathbb{R}^n$.*

until one concludes that x belongs to some family of uniformly bounded signals $\mathcal{X}_n$ with the desired asymptotic properties. A result of the **PD** type corresponds to a property of the discrete-logic

$$\sigma = \Phi(x, \sigma^-), \qquad\qquad t \geq 0, \tag{4}$$

whereas a result of the **PC** type corresponds to a property of the continuous-time switched system

$$\dot{x} = A_\sigma(x, d).$$

In the following sections we present several results of these types that are available in the literature. Section 2 focus on **PC** results, whereas Section 3 concentrates on **PD** results. Many of these lead directly to hybrid controller design methodologies. This is illustrated in Section 4 through two case studies.

For lack of space, we do not pursue analysis techniques based on *impact* or *Poicaré return maps*. The basic idea behind impact maps is to "sample" the continuous state at switching times and then analyze its evolution as if one was dealing with a discrete-time system. The main difficulty with this type of approach is that, because the sampling is not uniform over time, even for simple continuous dynamics (e.g., linear or affine), the "sampled" system may be very nonlinear and it may even be difficult to write it explicitly. However, this type of technique was used successfully, e.g., by Grizzle *et al.* (2001) to analyze bipedal walking robots and by Gonçalves *et al.* (2001) to analyze relay feedback systems.

2. Switched Systems

In this section we study the properties of a continuous-time switched system of the form

$$\dot{x} = A_\sigma(x, d), \qquad\qquad x \in \mathbb{R}^n,\ d \in \mathbb{R}^k, \tag{5}$$

where the family of vector fields $\{A_q(\cdot) : q \in \mathcal{Q}\}$ is given and the switching signal $\sigma : [0, \infty) \to \mathcal{Q}$ is known to belong to some set $\mathcal{S}$ of piecewise-constant signals.

We recall that $\mathcal{K}$ denotes the set of all continuous functions $\alpha : [0, \infty) \to [0, \infty)$ that are zero at zero, strictly increasing, and continuous; $\mathcal{K}_\infty$ the subset of $\mathcal{K}$ consisting of those functions that are unbounded; and $\mathcal{KL}$ the set of continuous functions $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ which, for every fixed value of the second argument, are of class $\mathcal{K}$ when regarded as functions of the first argument, and that have $\lim_{\tau\to\infty} \beta(s, \tau) = 0$ for every fixed $s \geq 0$. Given a vector $x \in \mathbb{R}^n$ we denote by $\|x\|$ the Euclidean norm of x.

We say that (5) is *uniformly asymptotically stable over* $\mathcal{S}$ if there exists a function β of class $\mathcal{KL}$ such that, for every $\sigma \in \mathcal{S}$,

$$\|x(t)\| \leq \beta(\|x(\tau)\|, t-\tau), \qquad \forall t \geq \tau \geq 0, \tag{6}$$

along solutions to (5) for which $d(t) = 0$, $t \geq 0$. When $\beta(s,t)$ is of the form $ce^{-\lambda t}s$ for some $c, \lambda > 0$ we say that (5) is *uniformly exponentially stable over* $\mathcal{S}$. In this case we can emphasize the rate of decay in the above bound by adding that (5) has *stability margin* λ. Local versions of these definitions can be obtained by restricting $x(\tau)$ in (6) to belong to an open neighborhood of the origin.

For exogenous inputs d that are not necessarily zero, we say that (5) is *uniformly input-to-state stable over* $\mathcal{S}$ if there exists a function α of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that, for every $\sigma \in \mathcal{S}$,

$$\|x(t)\| \leq \beta(\|x(\tau)\|, t-\tau) + \sup_{s\in[\tau,t)} \alpha(\|d(t)\|), \qquad \forall t \geq \tau \geq 0, \tag{7}$$

along solutions to (5). Replacing the $\sup_{s\in(t-\tau)}$ in (7) by the integral $\int_\tau^t \cdot ds$ over the same interval, we obtain the definition of *uniform integral-input-to-state stability over* $\mathcal{S}$.

When all the vector fields $A_q(\cdot)$, $q \in \mathcal{Q}$ are linear we say that (5) is a *linear switched system.* In case the set of matrices that represent these maps in some basis of $\mathbb{R}^n$ is compact, (5) is called a *compact linear switched system.* Compactness is automatically guaranteed whenever $\mathcal{Q}$ is finite. For compact linear systems, one can use fairly standard results to prove that uniform asymptotic stability is equivalent to uniform exponential stability (cf., e.g., the work of Molchanov and Pyatnitskiy, 1989, for details).

Similar to what happens for unswitched linear systems, uniform exponential stability of a compact linear switched system over $\mathcal{S}$ implies uniform input-to-state and integral-input-to-state stability over the same set $\mathcal{S}$. In fact, uniform exponential stability over $\mathcal{S}$, actually implies that several induced norms of (5) are uniformly bounded over $\mathcal{S}$. We define some of these norms next: Given a positive constant λ, we say that (5) has *input-to-state* $e^{\lambda t}$*-weighted,* L_∞*-induced norm uniformly bounded over* $\mathcal{S}$ if there exist finite constants g, g_0 such that, for every piecewise continuous input d and every $\sigma \in \mathcal{S}$,

$$e^{\lambda t}\|x(t)\| \leq g_0\, e^{\lambda\tau}\|x(\tau)\| + g \sup_{[\tau,t)} e^{\lambda s}\|d(s)\|, \qquad t \geq \tau \geq 0. \tag{8}$$

In general, this is stronger than uniform input-to-state stability because (8) implies (7) with $\beta(s,t) = g_0 e^{-\lambda t}s$ and $\alpha(s) = gs$, $t, s \geq 0$. When (8) is replaced by

$$e^{\lambda t}\|x(t)\| \leq g_0\, e^{\lambda\tau}\|x(\tau)\| + g\left(\int_0^t e^{2\lambda s}\|d(s)\|^2 ds\right)^{\frac{1}{2}}, \qquad t \geq \tau \geq 0, \tag{9}$$

we say that (5) has *input-to-state* $e^{\lambda t}$*-weighted,* L_2*-to-*L_∞*-induced norm uniformly bounded over* $\mathcal{S}$. In general, this is stronger than uniform integral-input-to-state stability because (9) implies that (7) holds with $\sup_{s\in(t-\tau)}$ replaced by $\int_\tau^t \cdot ds$, $\beta(s,t) = g_0 e^{-\lambda t}s$, and $\alpha(s) = gs$, $t, s \geq 0$. To verify that this is true one needs to use the fact that $(\int_a^b x^2)^{\frac{1}{2}} \leq \int_a^b |x|$ for every signal x for which the integrals exist. Finally, if (8) is replaced by

$$\left(\int_0^t e^{2\lambda\tau}\|x(\tau)\|^2\right)^{\frac{1}{2}} \leq g_0\|x(0)\| + g\left(\int_0^t e^{2\lambda\tau}\|d(\tau)\|^2\right)^{\frac{1}{2}}, \qquad t \geq 0, \tag{10}$$

we say that (5) has *input-to-state* $e^{\lambda t}$*-weighted,* L_2*-induced norm uniformly bounded over* $\mathcal{S}$. It is straightforward to show (cf., e.g., Hespanha and Morse, 1999b) that the following holds.

Lemma 1. *Suppose that* (5) *is a compact linear switched system. Given a family* $\mathcal{S}$ *of piecewise constant switching signals, if* (5) *is uniformly exponentially stable over* $\mathcal{S}$*, with stability margin* λ_0*, then, for every* $\lambda \in [0, \lambda_0)$*,* (5) *has input-to-state* $e^{\lambda t}$*-weighted,* L_∞*-induced norm uniformly bounded over* $\mathcal{S}$*. Similarly for the* L_2 *and* L_2*-to-*L_∞ *induced norms.*

The computation of L_2-induced norms for switched linear systems was studied by Hespanha (2002), which showed that even for very slow switching the induced norm of a switched system can be strictly larger than the norms of the systems being switched. In fact, the induced norm of a switched system is realization dependent and cannot be determined just from the transfer functions of the systems being switched.

We proceed to analyze the uniform stability of switched systems over several classes of switching signal.

2.1. Stability under Arbitrary Switching

We start by studying the stability of (5) over the set $\mathcal{S}_{\text{all}}$ of all piecewise continuous switching signals. A straightforward argument to prove uniform asymptotic stability over $\mathcal{S}_{\text{all}}$ is based on the existence of a *common Lyapunov function* V for the family of systems $\{\dot{z} = A_q(z, 0) : q \in \mathcal{Q}\}$, i.e., a continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ for which

$$\frac{\partial V}{\partial z}(z) A_q(z, 0) \leq W(z), \qquad \forall q \in \mathcal{Q},\ z \in \mathbb{R}^n \tag{11}$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$. We recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *radially unbounded* if $\lim_{z\to\infty} V(z) = \infty$, and V is said to be *positive (negative) definite* if, for all $z \in \mathbb{R}^n$, $V(z) \geq 0$ ($V(z) \leq 0$) with equality only at $z = 0$. It is well known from standard Lyapunov stability theory that, for any given $\sigma \in \mathcal{S}$, the existence of the Lyapunov function V for the time-varying system (5) guarantees that the origin is a globally asymptotically uniformly stable equilibrium point (cf. *Stability Theory*). Moreover, the inequality (6) holds for an appropriately defined function of class $\mathcal{KL}$ that can be constructed solely from V and W and that is therefore independent of σ (Theorem 3.8 Khalil, 1992). The following can then be stated:

Theorem 1. *If there exists a common Lyapunov function for the family of systems* $\{\dot{z} = A_q(z, 0) : q \in \mathcal{Q}\}$*, then the switched system* (5) *is uniformly asymptotically stable over* $\mathcal{S}_{\text{all}}$*.*

An analogous argument can be made for uniform input-to-state stability over $\mathcal{S}_{\text{all}}$ if we now require the existence of a *common ISS-Lyapunov function* V for the family of systems $\{\dot{z} = A_q(z, d) : q \in \mathcal{Q}\}$, i.e., a continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ for which

$$\|z\| \geq \gamma(\|d\|) \quad \Rightarrow \quad \frac{\partial V}{\partial z}(z) A_q(z, d) \leq W(z), \qquad \forall q \in \mathcal{Q},\ z \in \mathbb{R}^n$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$ and some class $\mathcal{K}$ function γ. The following theorem can be obtained by adapting the previous argument to the input-to-state framework introduced by Sontag (1989):

Theorem 2. *If there exists a common ISS-Lyapunov function for the family of systems* $\{\dot{z} = A_q(z, d) : q \in \mathcal{Q}\}$*, then the switched system* (5) *is uniformly input-to-state stable over* $\mathcal{S}_{\text{all}}$*.*

The existence of a common Lyapunov function may seem a too strong requirement to prove uniform stability of a switched system over $\mathcal{S}_{\text{all}}$. However, Molchanov and Pyatnitskiy (1989) showed that it is actually necessary for the uniform stability of linear switched systems. Dayawansa and Martin (1999) and Mancilla-Aguilar and García (2000) later extended this result to nonlinear switched systems. We summarize these results in the following theorem. We recall that a family of functions $\{f_q : q \in \mathcal{Q}\}$ defined on $\mathbb{R}^n$ is called *equibounded* if $\sup_{q\in\mathcal{Q}} \|f_q(z)\| < \infty$, for every $z \in \mathbb{R}^n$. The same family is called *uniformly locally Lipschitz* if for every compact subset $\mathcal{C}$ of $\mathbb{R}^n$, there exists a finite constant k such that $\|f_q(z_1) - f_q(z_2)\| \le k\|z_1 - z_2\|$, for all $z_1, z_2 \in \mathcal{C}$, $q \in \mathcal{Q}$.

Theorem 3. *Suppose that the family of functions $\{A_q(\cdot, 0) : q \in \mathcal{Q}\}$ is equibounded and uniformly locally Lipschitz. The switched system* (5) *is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$ if and only if there exists a common Lyapunov function V for the family of systems $\{\dot{z} = A_q(z, 0) : q \in \mathcal{Q}\}$. Moreover, if* (5) *is a compact linear switched system, then the common Lyapunov function V can be chosen strictly convex and homogeneous of degree two of a quasi-quadratic form, i.e., $V(z) = z^T P(z) z$, $z \in \mathbb{R}^n$, with $P(z) = P^T(z) = P(\lambda z)$, $\lambda \ge 0$, $z \in \mathbb{R}^n$.*

Although the common Lyapunov function can be chosen quasi-quadratic it cannot be chosen quadratic. In fact, Dayawansa and Martin (1999) provided a two-dimensional linear switched system (with switching just between two vector fields) that is uniformly exponentially stable over $\mathcal{S}_{\text{all}}$ but for which there is no common quadratic Lyapunov function.

Gurvits (1996) formulated another necessary and sufficient condition for the uniform asymptotic stability over $\mathcal{S}_{\text{all}}$ of a compact linear switched system. This conditions is expressed in terms of a sub-multiplicative matrix norm for which the state transition matrix $\Phi_q(t, \tau)$ of every system $\dot{z} = A_q(z, 0)$ can be norm-bounded by $e^{-\lambda(t-\tau)}$ for some $\lambda > 0$. We recall that a norm $\|\cdot\|_*$ defined on $\mathbb{R}^{n\times n}$ is called *sub-multiplicative*, if $\|AB\|_* \le \|A\|_* \|B\|_*$ for all $A, B \in \mathbb{R}^{n\times n}$. This result can be summarized as follows:

Theorem 4. *Suppose that* (5) *is a compact linear switched system. The switched system* (5) *is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$ if and only if there exists a sub-multiplicative norm $\|\cdot\|_*$ on $\mathbb{R}^{n\times n}$ and a positive constant λ such that*

$$\|\Phi_q(t, 0)\|_* \le e^{-\lambda t}, \qquad t \ge 0,\ q \in \mathcal{Q}, \tag{12}$$

where $\Phi_q(t, \tau)$ denotes the state transition matrix of the time-invariant linear system $\dot{z} = A_q(z, 0)$.

The sufficiency of (12) for uniform asymptotic stability over $\mathcal{S}_{\text{all}}$ is straightforward. Indeed, if we denote by $\{t_1, t_2, \dots, t_k\}$ the switching times in the interval (τ, t), then

$$x(t) = \Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \cdots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(\tau)}(t_1, \tau)x(\tau). \tag{13}$$

Because of (12) and the sub-multiplicative property of $\|\cdot\|_*$ we conclude that

$$\|\Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \cdots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(\tau)}(t_1, \tau)\|_* \le e^{-\lambda(t-\tau)}.$$

Since all norms of a finite dimensional space are equivalent we conclude that

$$\|\Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \cdots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(\tau)}(t_1, \tau)\|_i \le c e^{-\lambda(t-\tau)},$$

for some constant $c > 0$, where now $\|\cdot\|_i$ denotes the norm of a $n \times n$ matrix, induced by the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$. From this and (13) we conclude that

$$\|x(t)\| \le c e^{-\lambda(t-\tau)} \|x(\tau)\|,$$

and uniform asymptotic stability over $\mathcal{S}_{\text{all}}$ follows. The proof that the existence of the sub-multiplicative norm $\|\cdot\|_*$ is actually necessary for stability is considerably more complex and the reader is referred to (Gurvits, 1996) for details.

Algebraic Conditions for Stability under Arbitrary Switching

In practice, the results on stability under arbitrary switching presented above are difficult to apply because there is no simple method to find a common Lyapunov function or a norm for which (12) holds. However, for linear switched systems, there are simple algebraic conditions that are sufficient (and sometimes also necessary) for uniform asymptotic stability over $\mathcal{S}_{\text{all}}$. We review some of these next. In most of this section, we restrict our attention to linear switched systems with zero input $d = 0$. With some abuse of notation, we denote by A_q the matrix representation of the linear function $z \mapsto A_q(z, 0)$, $q \in \mathcal{Q}$ in the canonical basis of $\mathbb{R}^n$. In this case, we can re-write (5) as

$$\dot{x} = A_\sigma x, \qquad x \in \mathbb{R}^n. \tag{14}$$

Molchanov and Pyatnitskiy (1989) derived a necessary and sufficient algebraic condition for uniform asymptotic stability over $\mathcal{S}_{\text{all}}$. To achieve this they considered a higher dimensional switched system whose set of solutions "contains" all the solutions of the original system. The advantage of working with the higher dimensional system is that it can be chosen to admit a common quadratic Lyapunov function. Molchanov and Pyatnitskiy (1989) defined the new switched system as

$$\dot{z} = \bar{A}_\sigma z, \tag{15}$$

where each matrix $\bar{A}_q \in \mathbb{R}^{m \times m}$, $q \in \mathcal{Q}$ is a solution to

$$MA_q = \bar{A}_q M,$$

for some full column rank matrix $M \in \mathbb{R}^{m \times n}$, $m \geq n$. With these $\bar{A}_q$, for any given solution x to (14), $z := Mx$ is a solution to (15). Moreover, since M is full rank, $x = (M^T M)^{-1} M^T z$ and therefore if (15) is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$ then so is (14). This means that, e.g., if $V(z) := z^T z$ is a common Lyapunov function for the family of systems $\{\dot{z} = \bar{A}_q z : q \in \mathcal{Q}\}$, then (14) must be uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$. Molchanov and Pyatnitskiy (1989) went further and proved that the existence of the matrix M above for which $V(z) := z^T z$ is a common Lyapunov function for $\{\dot{z} = \bar{A}_q z : q \in \mathcal{Q}\}$ is not only sufficient, but also necessary for the uniformly asymptotic stability of (14) over $\mathcal{S}_{\text{all}}$. Their result can be stated as follows:

Theorem 5. *The switched system* (14) *is uniformly asymptotically stable over* $\mathcal{S}_{\text{all}}$ *if and only if there exists a full column rank matrix* $M \in \mathbb{R}^{m \times n}$, $m \geq n$, *and a family of matrices* $\{\bar{A}_q \in \mathbb{R}^{m \times m} : q \in \mathcal{Q}\}$ *that are a solution to*

$$MA_q = \bar{A}_q M, \qquad q \in \mathcal{Q},$$

for which $V(z) := z^T z$ *is a common Lyapunov function for the family of systems* $\{\dot{z} = \bar{A}_q z : q \in \mathcal{Q}\}$.

Unfortunately, applying Theorem 5 is still difficult because, in general, the numerical search for the matrix M is not simple.

In an attempt to find simple procedures to determined the stability of switched systems, several researchers restricted their attention to finding common quadratic Lyapunov functions. As mentioned above, the existence of a common quadratic Lyapunov function, although sufficient, is not necessary for uniform stability and therefore this restriction necessarily leads to some conservativeness.

Finding a common quadratic Lyapunov function $V(z) = z^T P z$, $z \in \mathbb{R}^n$ for the family of systems $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$, amounts to finding a solution to the following system of Linear Matrix Inequalities (LMIs) on $P = P^T \in \mathbb{R}^{n \times n}$ and $\epsilon \in \mathbb{R}$:

$$P > 0, \qquad A_q^T P + P A_q \leq -\epsilon I < 0, \qquad q \in \mathcal{Q}. \tag{16}$$

This condition is appealing because the feasibility of (16) can be determined numerically quite efficiently for finite $\mathcal{Q}$.

For certain classes of systems, it is even possible to verify directly that a common quadratic Lyapunov function exists. This happens, e.g., when all the matrices A_q, $q \in \mathcal{Q}$ are upper triangular or they are all lower triangular. Using straightforward linear algebra, it is not hard to verify that in this case one can choose a common Lyapunov function $V(z) = z^T P z$, $z \in \mathbb{R}^n$, with P diagonal (the algebraic derivations can be found, e.g., in the work of Yoshihiro Mori and Kuroe, 1997; Liberzon *et al.*, 1999; Shorten *et al.*, 1999). Compactness of the linear switched system is required in these derivations.

Although it may be unlikely that all the matrices A_q, $q \in \mathcal{Q}$ are in upper (lower) triangular form, it sometimes happens that these matrices can be put in this form by a *common similarity transformation*. Suppose that there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ for which all the matrices $T^{-1} A_q T$, $q \in \mathcal{Q}$ are upper (lower) triangular and let $V(z) = z^T P z$, $z \in \mathbb{R}^n$ be a common Lyapunov function for $\{\dot{z} = T^{-1} A_q T z : q \in \mathcal{Q}\}$. In this case

$$P > 0, \qquad T^T A_q^T T^{-T} P + P T^{-1} A_q T \leq -\epsilon I < 0, \qquad q \in \mathcal{Q}.$$

Left and right-multiplications of the above inequalities by T^{-T} and T^{-1}, respectively, lead to

$$Q > 0, \qquad A_q^T Q + Q A_q \leq -\epsilon I < 0, \qquad q \in \mathcal{Q},$$

where $Q := T^{-T} P T^{-1}$. This means that $V(z) = z^T Q z$, $z \in \mathbb{R}^n$ is a common Lyapunov function for $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$ and therefore (14) is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$. This was observed independently by Yoshihiro Mori and Kuroe (1997); Liberzon *et al.* (1999); Shorten *et al.* (1999). Yoshihiro Mori and Kuroe (1997); Liberzon *et al.* (1999) extended the above construction for similarity transformations T that are complex-valued, in which case the Q above would not necessarily be real-valued. These results can be summarized as follow:

Theorem 6. *Suppose that* (14) *is a compact linear switched system. If all the matrices A_q, $q \in \mathcal{Q}$ are asymptotically stable and there exists a nonsingular matrix $T \in \mathbb{C}^{n \times n}$ such that every matrix $\bar{A}_q := T^{-1} A_q T$, $q \in \mathcal{Q}$ is upper (lower) triangular, then there exists a common quadratic Lyapunov function for the family of systems $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$.*

The above result is particularly powerful because there is a necessary and sufficient algebraic condition for the existence of a similarity transformation that simultaneously upper triangularizes a given set of matrices. This condition is formulated in terms of the Lie algebra $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ generated by the matrices A_q. To proceed we need to introduce some definitions. Given a Lie algebra g, the descending sequence of ideals $g^{(k)}$ is defined inductively as follows: $g^{(1)} := g$, $g^{(k+1)} := [g^{(k)}, g^{(k)}] \subset g^{(k)}$. If $g^{(k)} = 0$ for k sufficiently large, then g is called *solvable*. Similarly, one defines the descending sequence of ideals g^k by $g^1 := g$, $g^{k+1} := [g, g^k] \subset g^k$, and calls g *nilpotent* if $g^k = 0$ for k sufficiently large. For example, if g is a Lie algebra generated by two matrices A and B, i.e., $g = \{A, B\}_{\text{LA}}$, then we have: $g^{(1)} = g^1 = g = \text{span}\{A, B, [A, B], [A, [A, B]], \dots\}$, $g^{(2)} = g^2 = \text{span}\{[A, B], [A, [A, B]], \dots\}$, $g^{(3)} = \text{span}\{[[A, B], [A, [A, B]]], \dots\} \subset g^3 = \text{span}\{[A, [A, B]], [B, [A, B]], \dots\}$, and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

Gurvits (1996) conjectured that if the Lie algebra $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ is nilpotent then (14) is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$. He used the Baker-Campbell-Hausdorff formula (Hilgert *et al.*, 1989) to prove this conjecture would hold for the particular case when $\mathcal{Q} = \{1, 2\}$, the matrices A_1 and A_2 are nonsingular, and their third-order Lie brackets vanish, i.e., $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$. The conjecture was actually formulated for discrete-time switched systems. However, the proof that was given for the special case mentioned above could easily be extended to continuous-time switched systems such as (14). Liberzon *et al.* (1999) proved that this conjecture was correct and that, in fact, even just solvability of $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ is sufficient for uniform stability. To understand why this is so we need to recall Lie's Theorem (cf., e.g., the standard textbook by Samelson, 1969).

Theorem 7 (Lie). *Let g be a Lie algebra over an algebraically closed field, and let ρ be a representation of g on a vector space V of finite dimension n. If g is solvable then there exists a basis $\{v_1, \dots, v_n\}$ of V such that for every $X \in g$ the matrix of $\rho(X)$ in that basis takes the upper triangular form*

$$\begin{bmatrix} \lambda_1(X) & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n(X) \end{bmatrix}$$

($\lambda_1(X), \dots, \lambda_n(X)$ being its eigenvalues). Conversely, the existence of a basis with the above property is sufficient for the solvability of g.

In our context, this means that if $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ is solvable then there exists a nonsingular complex matrix $T \in \mathbb{C}^{n\times n}$ such that $\bar{A}_q := T^{-1}A_qT$, $q \in \mathcal{Q}$ is upper triangular. Liberzon *et al.* (1999) observed that this could be combined with Theorem 6 to conclude the following:

Theorem 8. *Suppose that* (14) *is a compact linear switched system. If all the matrices A_q, $q \in \mathcal{Q}$ are asymptotically stable and the Lie algebra $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ is solvable then there exists a common quadratic Lyapunov function for the family of systems $\{\dot{z} = A_q : q \in \mathcal{Q}\}$.*

When all the matrices A_q, $q \in \mathcal{Q}$ commute pairwise, i.e.,

$$[A_{q_1}, A_{q_2}] := A_{q_1}A_{q_2} - A_{q_2}A_{q_1} = 0, \qquad\qquad q_1, q_2 \in \mathcal{Q}$$

then the Lie algebra $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ is nil-potent and therefore solvable. A common quadratic Lyapunov function is then guaranteed to exist. This result was proved by Narendra and Balakrishnan (1994) using a constructive procedure to directly generate the common Lyapunov function. It is not clear if Theorem 8 generalizes to nonlinear systems. However, Mancilla-Aguilar (2000) showed that the special case of vectors fields that commute pairwise leads to uniform stability for nonlinear switched systems:

Theorem 9. *If all the systems $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ are globally asymptotically stable and, for every $q_1, q_2 \in \mathcal{Q}$,*

$$\big[A_{q_1}(\cdot, 0), A_{q_2}(\cdot, 0)\big](z) := \frac{\partial A_{q_1}(z,0)}{\partial z}A_{q_2}(z, 0) - \frac{\partial A_{q_2}(z,0)}{\partial z}A_{q_1}(z, 0) = 0, \qquad \forall z \in \mathbb{R}^n,$$

then the switched system (14) *is uniformly asymptotically stable over $\mathcal{S}_{\text{all}}$. Moreover, if the family of functions $\{A_q(\cdot, 0) : q \in \mathcal{Q}\}$ is equibounded and uniformly locally Lipschitz then there exists a common Lyapunov function for the family of systems $\{\dot{z} = A_q(z, 0) : q \in \mathcal{Q}\}$. These two additional conditions hold trivially when the set $\mathcal{Q}$ is finite.*

Theorems 6 and 8 provide sufficient conditions for the existence of a common quadratic Lyapunov function for compact linear switched systems. It turns out that these results are conservative and far from necessary. In fact, solvability of the Lie algebra $\{A_q : q \in \mathcal{Q}\}_{\text{LA}}$ (and consequently simultaneous triangularization) is a fragile property in the sense that it can be lost with an arbitrarily small perturbation on the matrices A_q, $q \in \mathcal{Q}$. However, the existence of a common quadratic Lyapunov function is a robust property because it cannot be lost with an arbitrarily small perturbation. In fact, if $V(z) = z^T P z$, $z \in \mathbb{R}^n$ is a common Lyapunov function for the family of systems $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$ that satisfies (16), it is straightforward to compute an upper bound on admissible perturbations to the A_q, $q \in \mathcal{Q}$ so that $V(z)$ remains to be a common Lyapunov function for the perturbed systems.

To reduce the conservativeness of the results above, Shorten and Narendra (1999) derived a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for a pair of second order linear systems. Their result is based on the concept of a matrix pencil. The *matrix pencil* generated by two matrices $A_1, A_2 \in \mathbb{R}^n$ is defined to be the convex-hull of A_1 and A_2, i.e., the set

$$\rho[A_1, A_2] := \big\{\lambda A_1 + (1-\lambda)A_2 : \lambda \in [0,1]\big\}.$$

The importance of the matrix pencil stems from the fact that, for every $\lambda \in [0,1]$, it is possible to "emulate" a trajectory of the system

$$\dot{z} = \big(\lambda A_1 + (1-\lambda)A_2\big)z,$$

by switching very rapidly between the two systems $\dot{z} = A_1\, z$ and $\dot{z} = A_2\, z$ and keeping $\sigma = 1$ for λ-percent of the time. This "emulation" can be made arbitrarily accurate by increasing the frequency of switching. Because of this, the asymptotic stability of all the matrices in the pencil $\rho[A_1, A_2]$ is a necessary condition for the uniform asymptotic stability of the linear switched system (14). It turns out that, in general, this is not a sufficient condition. However, Shorten and Narendra (1999) derived the following necessary and sufficient condition for the existence of common quadratic Lyapunov functions for two second order linear systems.

Theorem 10. *The pair of systems* $\{\dot{z} = A_1\, z,\ \ \dot{z} = A_2\, z\}$, $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *has a common quadratic Lyapunov function if and only if all the matrices in the pencils* $\rho[A_1, A_2]$ *and* $\rho[A_1, A_2^{-1}]$ *are asymptotically stable.*

Under some technical assumptions, Shorten and Narendra (2000) generalized this result for any finite number of second order linear systems. However, the generalization for higher order systems remains elusive.

Controller realizations for Stability under Arbitrary Switching

The existence of a common quadratic Lyapunov function for a family of closed-loop systems $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$ depends on the realizations of the processes and controllers. When all the A_q, $q \in \mathcal{Q}$ are obtained from the interconnection of a single linear time-invariant process with several linear time-invariant candidate controllers, one can ask if it is possible to select realizations for the process and the candidate controllers so that a common quadratic Lyapunov function exists. We recall that a quadruple of matrices $\{A, B, C, D\}$ is called a *realization* for a transfer matrix T if $T(s) = C(sI - A)^{-1}B + D$ for every $s \in \mathbb{C}$. When the matrix D is equal to zero one often simply writes that $\{A, B, C\}$ is a realization for T.

Hespanha and Morse (2002) showed that the answer to the question above is affirmative, provided that all the candidate controllers asymptotically stabilize the process. This result can be summarized as follows:

Theorem 11. *Assume given a strictly proper process transfer matrix H and a family of controller transfer matrices $\mathcal{K}_C := \{K_q : q \in \mathcal{Q}\}$ such that every element of $\mathcal{K}_C$ asymptotically stabilizes H. There always exist realizations $\{F_q, G_q, H_q, J_q\}$ for each transfer matrix K_q in $\mathcal{K}_C$ such that, for an appropriate realization $\{A, B, C\}$ for H, there exists a common quadratic Lyapunov function for the family of systems $\{\dot{z} = A_q z : q \in \mathcal{Q}\}$, where each*

$$\dot{z} = A_q z,$$

denotes the feedback connection of the process realized by $\{A, B, C\}$ with the controller realized by $\{F_q, G_q, H_q, J_q\}$.

It should be noted that, in general, the realizations $\{F_q, G_q, H_q, J_q\}$ are not minimal. The proof of Theorem 11 by Hespanha and Morse (2002) is constructive and makes use of the Youla *et al.* (1976) parameterization of all controllers that stabilize H.

2.2. Stability under Slow Switching

Because the set $\mathcal{S}_{\text{all}}$ is very large, often one does not have uniform stability over this set. When this happens one is forced to consider "smaller" sets of switching signals. In this section we consider subsets of $\mathcal{S}_{\text{all}}$ for which the number of switchings on any given finite time interval is limited.

Given a positive constant τ_D, let $\mathcal{S}[\tau_D]$ denote the set of all switching signals with interval between consecutive discontinuities no smaller than τ_D. The constant τ_D is called the *(fixed) dwell-time*. It turns out that, when (5) is a compact linear switched system and all the systems $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ are asymptotically stable, then (5) is uniformly asymptotically stable, provided that τ_D is sufficiently large, i.e., that switching is sufficiently slow. To understand why this is so, first note that because of asymptotic stability and compactness there exist positive constants μ, λ_0 such that

$$\|\Phi_q(t, \tau)\|_i \leq \mu e^{-\lambda_0(t-\tau)}, \qquad t \geq \tau \geq 0,\ q \in \mathcal{Q}, \tag{17}$$

where $\Phi_q(t, \tau)$ denotes the state transition matrix of $\dot{z} = A_q(z, 0)$ and $\|\cdot\|_i$ the norm of a $n \times n$ matrix, induced by the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$. The constant λ_0 can be viewed as a common stability margin for all the $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$. Denoting by $\{t_1, t_2, \ldots, t_k\}$ the switching times in the interval (τ, t), then

$$x(t) = \Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \cdots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(\tau)}(t_1, \tau)x(\tau). \tag{18}$$

Assuming that $\sigma \in \mathcal{S}[\tau_D]$ then, for every $i \in \{2, 3, \ldots, k\}$,

$$\|\Phi_q(t_i, t_{i-1})\|_i \leq \mu e^{-\lambda_0(t_i - t_{i-1})} \leq \mu e^{-\lambda_0 \tau_D},$$

where $q := \sigma(t_{i-1})$. To have asymptotic stability it is sufficient to have $\mu e^{-\lambda_0 \tau_D} < 1$, which can be achieved with

$$\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}$$

for some $\lambda \in (0, \lambda_0)$. In this case

$$\|\Phi_q(t_i, t_{i-1})\|_i \leq \mu e^{-\lambda_0(t_i - t_{i-1})} \leq e^{(\lambda_0 - \lambda)\tau_D} e^{-\lambda_0(t_i - t_{i-1})} = e^{-(\lambda_0 - \lambda)(t_i - t_{i-1} - \tau_D) - \lambda(t_i - t_{i-1})} \leq e^{-\lambda(t_i - t_{i-1})}.$$

From this and (18) we conclude that (5) is uniformly exponentially stable over $\mathcal{S}[\tau_D]$ with stability margin λ. The following was proved:

Theorem 12. *Suppose that* (5) *is a compact linear switched system and that* λ_0 *is a positive constant such that all the linear systems* $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ *are exponentially stable with stability margin* λ_0. *For every* $\lambda \in (0, \lambda_0)$, (5) *is uniformly exponentially stable over* $\mathcal{S}[\tau_D]$ *with stability margin* λ, *for any fixed dwell-time* $\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}$.

Hespanha and Morse (1999b) showed that a more general result still holds when $\mathcal{S}[\tau_D]$ is enlarged to contain signals that occasionally have consecutive discontinuities separated by less than τ_D, but for which the average interval between consecutive discontinuities is no less than τ_D. We proceed to formalize this concept of "average dwell-time." For each switching signal σ and each $t \geq \tau \geq 0$, let $N_\sigma(t, \tau)$ denote the number of discontinuities of σ in the open interval (τ, t). For given $N_0, \tau_D > 0$, we denote by $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ the set of all switching signals for which

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_D}, \qquad \forall t \geq \tau \geq 0.$$

The constant τ_D is called the *average dwell-time* and N_0 the *chatter bound*. In essence, a switching signal has finite average dwell-time if the number of discontinuities it exhibits in any given interval grows no larger than linearly with the length of the interval. For a signal to have average dwell-time τ_D, the growth rate must be no larger than $1/\tau_D$. Note that $\mathcal{S}[\tau_D] = \mathcal{S}_{\text{ave}}[\tau_D, 1]$. Going back to (18) and using (17), we conclude that

$$\|\Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \cdots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(\tau)}(t_1, \tau)\|_i \leq \mu^{k+1} e^{-\lambda_0(t-\tau)}.$$

To have exponential stability with stability margin λ, we need to have

$$\mu^{k+1} e^{-\lambda_0(t-\tau)} \leq c e^{-\lambda(t-\tau)}, \tag{19}$$

for some positive constant c, or, equivalently,

$$k \leq \frac{\log c}{\log \mu} - 1 + \frac{\lambda_0 - \lambda}{\log \mu}(t - \tau).$$

Since k is precisely the number of discontinuities of σ in (τ, t), we conclude that exponential stability is guaranteed provided that $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$ for every average dwell-time $\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}$. Any chatter bound N_0 can be accommodated at the expense of having a sufficiently large constant c in the bound (19). The following was proved:

Theorem 13. *Suppose that* (5) *is a compact linear switched system and that* λ_0 *is a positive constant such that all the linear systems* $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ *are exponentially stable with stability margin* λ_0. *For every* $\lambda \in (0, \lambda_0)$, (5) *is uniformly exponentially stable over* $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ *with stability margin* λ, *for any average dwell-time* $\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}$ *and any chatter bound* $N_0 > 0$.

This result can be extended to certain classes of nonlinear switched systems: For (5) to be uniformly asymptotically stable over the set $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, $\tau_D, N_0 > 0$ the origin must be a globally asymptotically stable equilibrium point of every time-invariant system $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$. Here we actually demand more of the $A_q(\cdot, 0)$:

Assumption 1. There exist continuously differentiable functions $V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ positive constants λ_0, μ, and functions $\alpha, \bar{\alpha}$ of class $\mathcal{K}_\infty$ such that

$$\frac{\partial V_q}{\partial z} A_q(z, 0) \leq -\lambda_0 V_q, \tag{20}$$

$$\alpha(\|z\|) \leq V_q(z) \leq \bar{\alpha}(\|z\|), \tag{21}$$

$$V_q(z) \leq \mu V_{\bar{q}}(z), \tag{22}$$

for every $z \in \mathbb{R}^n$ and $q, \bar{q} \in \mathcal{Q}$.

Equations (20)–(21) are the standard conditions for V_q to be a Lyapunov function of $\dot{z} = A_q(z, 0)$. It should be noted that, in light of the results by Sontag and Wang (1996); Praly and Wang (1996), the exponential decay suggested by (20) does not really introduce loss of generality. As for (22), it may, in fact, reduce the class of systems to which the results in this section can be applied. Equation (22) can be violated, for example, if the value of the parameter changes significantly the nature of the Lyapunov function, e.g., $\mathcal{Q} = \{1, 2\}$ and

$$V_q(x) = \begin{cases} x^2 & q = 1 \\ x^4 & q = 2 \end{cases}.$$

Under the above assumptions the following generalization of Theorem 13 is possible:

Theorem 14. *When Assumptions 1 holds, the switched system* (5) *is uniformly asymptotically stable over* $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, *for any average dwell-time* $\tau_D > \frac{\log \mu}{\lambda_0}$ *and any chatter bound* $N_0 > 0$.

A proof of this theorem can be constructed much like that of Theorem 13, except that now one shows that $v(t) := V_{\sigma(t)}\big(x(t)\big)$, $t \geq 0$ decays exponentially fast, instead of $\|\Phi(t, \tau)\|_i$, as was done before.

2.3. Stability under State-dependent Switching

So far we have studied the stability of (5) over sets of switching signals that were defined without regard to the evolution of the state $x(t)$ of (5). In general, this leads to conservative requirements on the vectors fields $A_q(\cdot)$, $q \in \mathcal{Q}$ being switched. For example, in all the cases above, stability of all the systems

$$\dot{z} = A_q(z, d), \qquad q \in \mathcal{Q},$$

was required for the uniform stability of the switched system. We consider next classes of switching signals for which this is no longer necessary. For simplicity, in this section we consider the case of a finite number of switched systems.

Suppose that for every possible value of the state $x(t)$ of (5), the switching signal $\sigma(t)$ is restricted to belong to a pre-specified subset of $\mathcal{Q}$. This can be formalized by the condition

$$\sigma(t) \in \mathcal{S}[x(t)], \qquad t \geq 0, \tag{23}$$

where each $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ is a nonempty subset of $\mathcal{Q}$. We abbreviate (23) by $\sigma \in \mathcal{S}[x]$ and say that σ *belongs to (the state-dependent family of switching signals)* $\mathcal{S}[x]$. Although $\mathcal{S}[x]$ is not really a set of switching signals, it is straightforward to extend the previous definitions of uniform stability to this type of switching. The constraint on σ posed by (23) can also be formalized by introducing a covering $\mathcal{X}_q$, $q \in \mathcal{Q}$ of $\mathbb{R}^n$ and requiring that

$$x(t) \in \mathcal{X}_{\sigma(t)}, \qquad t \geq 0. \tag{24}$$

This condition can be abbreviated by $x \in \mathcal{X}_\sigma$. The conditions (23) and (24) are equivalent, provided that we choose

$$\mathcal{X}_q := \{z \in \mathbb{R}^n : q \in \mathcal{S}[z]\}, \qquad q \in \mathcal{Q}. \tag{25}$$

Since all the $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ are nonempty, the sets $\mathcal{X}_q$, $q \in \mathcal{Q}$ defined in (25) indeed form a covering of $\mathbb{R}^n$. Figure 2 shows a covering $\mathcal{X}_q$, $q \in \mathcal{Q} := \{1, 2, 3\}$ of $\mathbb{R}^2$ and the corresponding sets $\mathcal{S}[z]$, $z \in \mathbb{R}^2$. In this section we mostly use (23) to specify state-dependent switching because it emphasizes the fact that the results presented are valid for restricted classes of

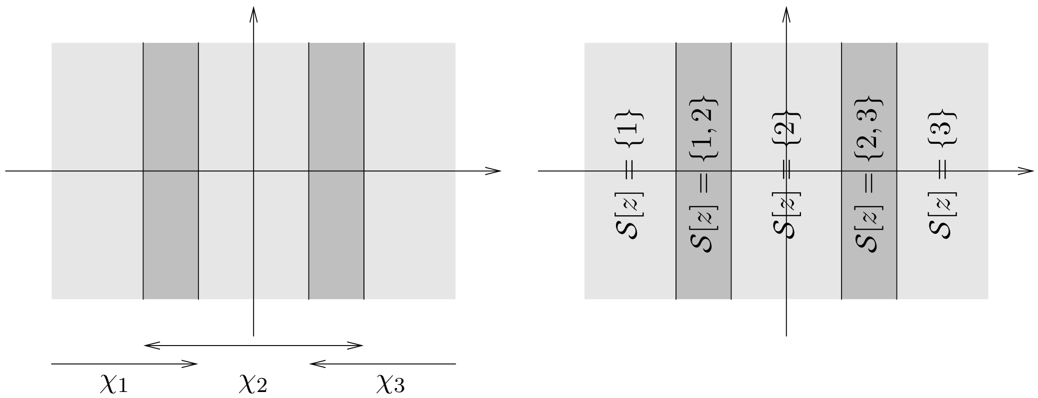

Figure 2: State-dependent switching

switching signals. However, in the hybrid systems literature it is also common to find state-dependent switching specified by expressions similar to (24). When σ is part of the discrete component of the state, (24) is often known as a guard condition.

Two important questions for the analysis and design of hybrid controllers are then:

> *Given a collection of systems $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ and nonempty sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$ is* (5) *uniformly asymptotically stable over $\mathcal{S}[x]$?*
>
> *Given a collection of systems $\dot{z} = A_q(z, 0)$, $q \in \mathcal{Q}$ is it possible to find nonempty sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$ that will make* (5) *uniformly asymptotically stable over $\mathcal{S}[x]$?*

In case the answer to the latter question is affirmative, in principle, it will be possible to design a hybrid supervisor that stabilizes the switched system. This supervisor simply has to generate σ so that (23) holds. We defer the details of the design of such a supervisor to Section 3 and focus now on answering the two questions formulated above.

State-dependent Common Lyapunov Function

A simple way to prove uniform asymptotic stability of (5) over $\mathcal{S}[x]$ is through the use of a *state-dependent common Lyapunov function for* $\mathcal{S}[x]$, i.e., a continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ for which

$$q \in \mathcal{S}[z] \quad \Rightarrow \quad \frac{\partial V}{\partial z}(z) A_q(z, 0) \leq W(z), \qquad\qquad q \in \mathcal{Q},\ z \in \mathbb{R}^n, \tag{26}$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$. The key difference with respect to the state-independent switching considered in Section 2.1 is that now we only require $\frac{\partial V}{\partial z}(z) A_q(z, 0)$ to be negative when σ may be equal to q, i.e., inside $\mathcal{S}[x(t)]$ (cf. (11)). Using an argument similar to the one used to prove Theorem 1 we conclude that

Theorem 15. *If there exists a state-dependent common Lyapunov function for $\mathcal{S}[x]$, then the switched system* (5) *is uniformly asymptotically stable over $\mathcal{S}[x]$.*

It should be emphasized that the existence of a state-dependent common Lyapunov function for some $\mathcal{S}[x]$ does *not* require all the systems $\dot{z} = A_q(z, 0)$ to be stable. In fact, a state-dependent common Lyapunov function for $\mathcal{S}[x]$ may exist even when all these systems are unstable.

In light of Theorem 15, Theorem 2 could also be adapted to the case of state-dependent switching. We leave this to the reader, as well as the generalization of the results that follow to systems with a nonzero input d.

We consider now the problem of constructing sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$ for which there exists a state-dependent common Lyapunov function. Suppose that there exist nonnegative constants $\lambda_q > 0$, $q \in \mathcal{Q}$ such that $\sum_{q\in\mathcal{Q}} \lambda_q = 1$ and the system

$$\dot{z} = \sum_{q\in\mathcal{Q}} \lambda_q A_q(z, 0), \tag{27}$$

is globally asymptotically stable. Using a converse Lyapunov function argument (e.g., Theorem 4.2 of the standard textbook by Hale, 1980), we conclude that there must exist a Lyapunov function to (27), i.e., a continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$, and a negative definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial z}(z) \sum_{q\in\mathcal{Q}} \lambda_q A_q(z, 0) \leq W(z). \tag{28}$$

Since, (28) is also equivalent to

$$\sum_{q\in\mathcal{Q}} \lambda_q \left(\frac{\partial V}{\partial z}(z) A_q(z, 0) - W(z) \right) \leq 0, \qquad \forall z \in \mathbb{R}^n,$$

we conclude that, for every $z \in \mathbb{R}^n$, at least one of the terms $\frac{\partial V}{\partial z}(z)A_q(z, 0) - W(z)$ is nonpositive and therefore the following set will be nonempty

$$\mathcal{S}[z] := \left\{ q \in \mathcal{Q} : \frac{\partial V}{\partial z}(z) A_q(z, 0) - W(z) \leq 0 \right\}.$$

Since (26) is automatically satisfied for this definition of $\mathcal{S}[z]$, we conclude the following:

Lemma 2. *If there exist positive constants $\lambda_q > 0$, $q \in \mathcal{Q}$ such that $\sum_{q\in\mathcal{Q}} \lambda_q = 1$ and (27) is globally asymptotically stable, then there exit nonempty sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$, and a state-dependent common Lyapunov function V for $\mathcal{S}[x]$.*

Feron (1996) showed that, for linear switched systems with $\mathcal{Q} := \{1, 2\}$, if we restrict V and W in (26) to be quadratic functions—a case known as *quadratic stabilizability*—the stability of (27) for some $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, is also a necessary condition for the existence of a state-dependent common Lyapunov function for some $\mathcal{S}[x]$. However, DeCarlo *et al.* (2000) noted that this result does not generalize to switching among more than two systems. It should also be mentioned that, even for linear systems, the verification of the hypothesis of Lemma 2 is not easy from a computational point of view. In fact, this problem is known to be NP-complete as shown by Blondel and Tsitsiklis (1997, 2000).

Multiple Lyapunov Functions

Even the existence of a state-dependent common Lyapunov function may be a very restrictive requirement. This can be alleviated through the introduction of multiple Lyapunov functions to analyze switched systems. These ideas were introduced in the hybrid systems community by Peleties and DeCarlo (1991); Branicky (1998). We say that a set $\{V_q : q \in \mathcal{Q}\}$ of continuously differentiable, radially unbounded, positive definite functions from $\mathbb{R}^n$ to $\mathbb{R}$ is a *family of multiple Lyapunov functions for $\mathcal{S}[x]$* if

$$q \in \mathcal{S}[z] \quad \Rightarrow \quad \frac{\partial V_q}{\partial z}(z) A_q(z, 0) \leq W(z), \qquad q \in \mathcal{Q},\ z \in \mathbb{R}^n,$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$. (We will see shortly that this definition can actually be relaxed.) The idea behind multiple Lyapunov functions is to piece them together to build a signal that provides a measure of the size of the state and is always decreasing. The natural candidate is

$$v(t) := V_{\sigma(t)}\big(x(t)\big), \tag{29}$$

because, between switching times,

$$\dot{v} = \frac{\partial V_q}{\partial x}(x)A_q(x,0) \le W\big(x\big) \le 0, \tag{30}$$

where q denotes the constant value of σ between switchings. However, when σ switches from a given value $q_1 \in \mathcal{Q}$ to another value $q_2 \in \mathcal{Q}$ at some time t, v may be discontinuous at t and even increase if the values of $V_{q_1}(x(t))$ and $V_{q_2}(x(t))$ are not equal. One method to guarantee stability of a switched system based on multiple-Lyapunov functions is to require that the multiple Lyapunov functions take the same value at any point where switching can occur. This requirement can be formulated as the following *matching condition*:

$$q_1, q_2 \in \bar{\mathcal{S}}[z] \quad \Rightarrow \quad V_{q_1}(z) = V_{q_2}(z), \qquad\qquad q_1, q_2 \in \mathcal{Q},\ z \in \mathbb{R}^n, \tag{31}$$

where $\bar{\mathcal{S}}[z]$ denotes the set of values that σ can take at points arbitrarily close to z, i.e.,

$$\bar{\mathcal{S}}[z] := \big\{q \in \mathcal{Q} : \forall \epsilon > 0, \exists \bar{z} : \|z - \bar{z}\| < \epsilon,\ q \in \mathcal{S}[\bar{z}]\big\} \supset \mathcal{S}[z].$$

The set $\bar{\mathcal{S}}[z]$ only contains multiple elements when z is at the boundary of one of the sets $\mathcal{X}_q$ defined in (25), or where these sets overlap.

Because the state x of (5) is continuous, if σ switches from $q_1 \in \mathcal{Q}$ to $q_2 \in \mathcal{Q}$ at some time t, we must have $q_1, q_2 \in \bar{\mathcal{S}}[x(t)]$ and therefore (31) guarantees that $V_{q_1}(x(t)) = V_{q_2}(x(t))$. This means that $v(t)$ in (29) is continuous. Using (30) we can then conclude stability of the switched system using standard arguments (that can be found, e.g., in the textbook by Khalil, 1992).

Actually, the functions V_q, $q \in \mathcal{Q}$ need not be positive definite and radially unbounded everywhere since $\sigma(t)$ can only be equal to q when $q \in \mathcal{S}[x(t)]$. This means that we can relax the definition of *multiple Lyapunov functions for* $\mathcal{S}[x]$ to be a family $\{V_q : q \in \mathcal{Q}\}$ of continuously differentiable functions for which there exists a function α of class $\mathcal{K}_\infty$ such that

$$q \in \mathcal{S}[z] \quad \Rightarrow \quad V_q(z) \ge \alpha(\|z\|),\ \ \frac{\partial V_q}{\partial z}(z)A_q(z,0) \le W(z), \qquad q \in \mathcal{Q},\ z \in \mathbb{R}^n. \tag{32}$$

The following can then be stated:

Theorem 16. *If $\{V_q : q \in \mathcal{Q}\}$ is a family of multiple Lyapunov functions for $\mathcal{S}[x]$ and the matching condition* (31) *holds, then the switched system* (5) *is uniformly asymptotically stable over $\mathcal{S}[x]$.*

We consider now the problem of constructing sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$, as well as multiple Lyapunov functions for $\mathcal{S}[x]$, for which the matching condition (31) holds. A common approach to solve this problem is to pick a particular structure for the sets $\mathcal{S}[z]$ and then numerically find a family of multiple Lyapunov functions. By proper choice of the $\mathcal{S}[z]$ it is often possible to make the search for the V_q numerically efficient. In the remaining of this section, we restrict our attention to linear switched systems with zero input $d = 0$. With some abuse of notation, we denote by A_q the matrix representation of the linear function $z \mapsto A_q(z, 0)$, $q \in \mathcal{Q}$ in the canonical basis of $\mathbb{R}^n$.

Inspired by the work of Wicks and Carlo (1997), we consider Lyapunov functions of the form

$$V_q(z) := z^T P_q z, \qquad\qquad z \in \mathbb{R}^n,$$

and sets $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ of the form

$$\mathcal{S}[z] := \{q \in \mathcal{Q} : V_q(z) \geq V_{\bar{q}}(z), \;\; \forall \bar{q} \in \mathcal{Q}\}. \tag{33}$$

Because of this definition, switching is only possible at points where the largest V_q's have the same value. We can then write $\bar{\mathcal{S}}[z]$ as

$$\bar{\mathcal{S}}[z] = \Big\{q \in \mathcal{Q} : V_q(z) = \max_{\bar{q}\in\mathcal{Q}} V_{\bar{q}}(z)\Big\}.$$

This automatically implies that the matching condition (31) holds because when $q_1, q_2 \in \bar{\mathcal{S}}[z]$, we have $V_{q_1}(z) = V_{q_2}(z) = \max_{\bar{q}\in\mathcal{Q}} V_{\bar{q}}(z)$. For the V_q, $q \in \mathcal{Q}$ to be multiple Lyapunov functions we can then simply require that

$$z^T P_q z \geq z^T P_{\bar{q}} z, \;\; \forall \bar{q} \in \mathcal{Q} \quad \Rightarrow \quad z^T P_q z \geq \delta\|z\|^2, \quad z^T(A_q^T P_q + P_q A_q) z \leq -\epsilon\|z\|^2, \;\; q \in \mathcal{Q}, \; z \in \mathbb{R}^n, \tag{34}$$

for some positive constants ϵ, δ. It is straightforward to verify that the following system of matrix inequalities is a sufficient condition for (34) to hold:

$$P_q - \sum_{\bar{q}\in\mathcal{Q}\setminus\{q\}} \gamma_{q,\bar{q}}(P_q - P_{\bar{q}}) > 0, \qquad A_q^T P_q + P_q A_q + \sum_{\bar{q}\in\mathcal{Q}\setminus\{q\}} \mu_{q,\bar{q}}(P_q - P_{\hat{q}}) < 0, \qquad q \in \mathcal{Q}, \tag{35}$$

where the $\gamma_{q,\bar{q}}$, $\mu_{q,\bar{q}}$, $q, \bar{q} \in \mathcal{Q}$ can be any nonnegative constants. Under certain conditions, (35) is actually equivalent to (34) (cf. $\mathcal{S}$-procedure described, e.g., in Boyd *et al.*, 1994). The following was proved

Lemma 3. *If either* (34) *or* (35) *hold for some symmetric matrices P_q, $q \in \mathcal{Q}$ then $\{V_q(z) := z^T P_q z : q \in \mathcal{Q}\}$ is a family of multiple Lyapunov functions for $\mathcal{S}[x]$, with the $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ defined by* (33)*. Moreover, the matching condition* (31) *holds.*

Johansson and Rantzer (1998) considered sets $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ that result from the following construction: They start by covering $\mathbb{R}^n$ with closed convex polyhedral cells $\mathcal{X}_q$, $q \in \mathcal{Q}$ with disjoint interiors and then only allow σ to take the value $q \in \mathcal{Q}$, while x is inside $\mathcal{X}_q$. Because the interior of the cells do not overlap, only at points $z \in \mathbb{R}^n$ on the boundaries of a cell, we can have more than one element in $\mathcal{S}[z]$ or $\bar{\mathcal{S}}[z]$. Since the cells are convex polyhedra, we can write $\mathcal{S}[z]$ as

$$\mathcal{S}[z] = \{q \in \mathcal{Q} : E_q \begin{bmatrix} z^T & 1 \end{bmatrix}^T \succeq 0\}, \qquad\qquad z \in \mathbb{R}^n, \tag{36}$$

for appropriately defined matrices $E_q \in \mathbb{R}^{n_E\times(n+1)}$, $q \in \mathcal{Q}$. Here, given two vectors $a, b \in \mathbb{R}^k$, we write $a \succeq b$ to mean that every entry of a is larger or equal to the corresponding entry of b. Moreover, the intersection between any two cells always occurs inside a $n-1$-dimensional affine subspace of $\mathbb{R}^n$. We can therefore find a set of vectors $f_{q_1q_2} \in \mathbb{R}^{n+1}$, $(q_1, q_2) \in \partial\mathcal{Q}$ such that

$$q_1, q_2 \in \bar{\mathcal{S}}[z] \quad \Rightarrow \quad f_{q_1q_2}^T \begin{bmatrix} z^T & 1 \end{bmatrix}^T = 0, \quad (q_1, q_2) \in \partial\mathcal{Q}. \tag{37}$$

Here, $\partial\mathcal{Q}$ denotes the set of pairs $(q_1, q_2) \in \mathcal{Q} \times \mathcal{Q}$ for which the cells $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$ have a common boundary. Suppose now that we restrict our attention to multiple Lyapunov functions of the form

$$V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z^T & 1 \end{bmatrix}^T, \qquad\qquad q \in \mathcal{Q}.$$

Because of (37), the matching condition (31) can be enforced by requiring that

$$f_{q_1 q_2}^T \bar{z} = 0 \quad \Rightarrow \quad \bar{z}^T P_{q_1} \bar{z} = \bar{z}^T P_{q_2} \bar{z}, \qquad (q_1, q_2) \in \partial\mathcal{Q},\ \bar{z} \in \mathbb{R}^{n+1}.$$

This is equivalent to the existence of vectors $k_{q_1 q_2} \in \mathbb{R}^{n+1}$, $(q_1, q_2) \in \partial\mathcal{Q}$ such that

$$P_{q_1} - P_{q_2} = k_{q_1 q_2}^T f_{q_1 q_2} + f_{q_1 q_2}^T k_{q_1 q_2}, \qquad (q_1, q_2) \in \partial\mathcal{Q} \tag{38}$$

(cf. Lemma 7 in the Appendix). On the other hand, to guarantee that the $\{V_q : q \in \mathcal{Q}\}$ are a family of multiple Lyapunov functions and therefore that (32) holds, it is sufficient to require that there exist positive constants ϵ, δ and symmetric matrices U_q, W_q, $q \in \mathcal{Q}$, with nonnegative entries (but not necessarily positive semi-definite) such that

$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi, \qquad \Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi, \qquad q \in \mathcal{Q}, \tag{39}$$

where $\Pi := \begin{bmatrix} I_n & 0_{1\times n} \end{bmatrix}$. This is because when $q \in \mathcal{S}[z]$ we have $E_q \begin{bmatrix} z^T & 1 \end{bmatrix}^T \succeq 0$ and therefore

$$V_q(z) = \begin{bmatrix} z^T & 1 \end{bmatrix} (P_q - E_q^T W_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} + \begin{bmatrix} z^T & 1 \end{bmatrix} E_q^T W_q E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq \delta \begin{bmatrix} z^T & 1 \end{bmatrix} \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = \delta \|z\|^2$$

and

$$\begin{aligned} \frac{\partial V_q}{\partial z}(z) A_q z &= \begin{bmatrix} z^T & 1 \end{bmatrix} (\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} - \begin{bmatrix} z^T & 1 \end{bmatrix} E_q^T U_q E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \\ &\leq -\epsilon \begin{bmatrix} z^T & 1 \end{bmatrix} \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = -\epsilon \|z\|^2. \end{aligned}$$

The following was proved:

Lemma 4. *Suppose that the $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ satisfy* (36)*. If* (38) *and* (39) *hold for appropriate vectors $k_{q_1 q_2} \in \mathbb{R}^{n+1}$, $(q_1, q_2) \in \partial\mathcal{Q}$, positive constants ϵ, δ, and symmetric matrices P_q, U_q, W_q, $q \in \mathcal{Q}$, with all the entries of the U_q and W_q nonnegative, then $\{V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z^T & 1 \end{bmatrix}^T : q \in \mathcal{Q}\}$ is a family of multiple Lyapunov functions for $\mathcal{S}[x]$. Moreover, the matching condition* (31) *holds.*

Since (39) forms a system of linear matrix inequalities on the unknowns $k_{q_1 q_2} \in \mathbb{R}^{n+1}$, $(q_1, q_2) \in \partial\mathcal{Q}$, $\epsilon, \delta, P_q, U_q, W_q$, $q \in \mathcal{Q}$, subject to the equality constraints (38), there are efficient numerical algorithms to search for these matrices (Boyd *et al.*, 1994).

Johansson and Rantzer (1998) actually considered an alternative formulation where the affine subspaces of $\mathbb{R}^n$ that contain the boundaries between the cells $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$, $q_1, q_2 \in \mathcal{Q}$ are defined by equations of the form

$$F_{q_1} \begin{bmatrix} z \\ 1 \end{bmatrix} = F_{q_2} \begin{bmatrix} z \\ 1 \end{bmatrix},$$

for appropriately defined matrices $F_q \in \mathbb{R}^{n_F \times (n+1)}$, $q \in \mathcal{Q}$. They then restricted their attention to multiple Lyapunov functions of the form

$$V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z \\ 1 \end{bmatrix}, \qquad q \in \mathcal{Q},$$

with $P_q := F_q^T S F_q$ for some symmetric matrix $S \in \mathbb{R}^{n_F \times n_F}$. This automatically implies that the matching condition (31) holds because when $q_1, q_2 \in \bar{\mathcal{S}}[z]$, the vector z must be at the boundary between $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$ and therefore

$$V_{q_1}(z) = \begin{bmatrix} z^T & 1 \end{bmatrix} F_{q_1}^T S F_{q_1} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} z^T & 1 \end{bmatrix} F_{q_2}^T S F_{q_2} \begin{bmatrix} z \\ 1 \end{bmatrix} = V_{q_2}(z).$$

This means that we could replace (38) by

$$P_q = F_q^T S F_q, \qquad q \in \mathcal{Q}$$

in Lemma 4. This formulation avoids the equality constraints (38) but requires the *a-priori* selection of the matrices F_q, $q \in \mathcal{Q}$. It turns out that, for the same collection of cells, many choices for the F_q are possible and many of them will lead to the unfeasibility of (39). The approach that led to Lemma 4 avoids this difficulty and leads to the least conservative set of conditions that enforce the matching condition over the affine subspaces of $\mathbb{R}^n$ that contain the cell boundaries.

State-dependent Switching with Constraints

So far we have assumed that we are given a collection of sets $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ such that $\sigma(t)$ can take any value in $\mathcal{S}[x(t)]$. We consider next a restricted class of switching signals for which we further require the discontinuities of σ to occur at specific locations. In particular, we take as given a collection of sets $\mathcal{D}[z] \subset \mathcal{Q} \times \mathcal{Q}$, $z \in \mathbb{R}^n$ and restrict σ to switch from $q_1 \in \mathcal{Q}$ to $q_2 \in \mathcal{Q}$ at time t only when the pair (q_1, q_2) belongs to $\mathcal{D}[x(t)]$, i.e.,

$$\big(\sigma^-(t), \sigma(t)\big) \in \mathcal{D}[x(t)], \tag{40}$$

for any switching time $t \geq 0$, where $\sigma^-(t)$ denotes the limit from the left of $\sigma(\tau)$ as $\tau \uparrow t$. This type of constraint on the switching signals occurs naturally when σ is generated by a discrete-logic such as (4). Note that some of the sets $\mathcal{D}[z]$ may be empty, which means that no switching can occur for those values of z. We abbreviate (23) and (40) by $\sigma \in \mathcal{S}[x] \cap \mathcal{D}[x]$ and say that σ belongs to the *state-dependent family of switching signals* $\mathcal{S}[x] \cap \mathcal{D}[x]$. Also here it is straightforward to extend the definitions of uniform stability to this type of switching.

The constraint posed by (40) restricts the regions of the state space where switching can occur and also the values to which σ can switch. When using multiple Lyapunov functions to prove stability, this allows us to significantly weaken the matching condition (31). This condition was used to guarantee that the signal v defined by (29) did not increase at switching times. Without (40), when $\bar{\mathcal{S}}[z]$ has more than one element—e.g., two elements q_1 and q_2—we can have switching from q_1 to q_2 or vice versa, so we required V_{q_1} and V_{q_2} to be equal at z so that v never increases. Now, to make sure that v does not increase, it is sufficient to require that

$$(q_1, q_2) \in \mathcal{D}[z] \quad \Rightarrow \quad V_{q_1}(z) \geq V_{q_2}(z), \qquad q_1, q_2 \in \mathcal{Q},\ z \in \mathbb{R}^n. \tag{41}$$

The following can now be stated:

Theorem 17. *If $\{V_q : q \in \mathcal{Q}\}$ is a family of multiple Lyapunov functions for $\mathcal{S}[x]$ and* (41) *holds, then the switched system* (5) *is uniformly asymptotically stable over $\mathcal{S}[x] \cap \mathcal{D}[x]$.*

Branicky (1998) derived generalizations of Theorem 17 specially tailored to prove the stability of specific classes of hybrid systems. We leave to the reader possible generalizations of Lemmas 3 and 4 to this setting. Some of these generalizations were proposed by Johansson and Rantzer (1998) and DeCarlo *et al.* (2000).

3. Supervisors

In this section we describe some of the supervisors that have been used in hybrid controllers and study their basic properties.

3.1. Dwell-time Supervisors

Dwell-time supervisors are used when one desires to enforce a particular evolution for the switching signal σ but, due to stability considerations, one needs to keep the number of switchings small. For example, given a family of sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$ one may want to have

$$\sigma(t) \in \mathcal{S}[x(t)], \qquad t \geq 0, \tag{42}$$

but also keep σ in the family $\mathcal{S}[\tau_D]$ of switching signals with interval between consecutive discontinuities no smaller than $\tau_D > 0$. Typically, $\sigma \in \mathcal{S}[\tau_D]$ is needed for the stability of the switched system, whereas (42) is desirable due to performance specifications. In this case, the former condition has priority over the second when they are not compatible.

Dwell-time supervisors resolve the conflicting requirements mentioned above by selecting a value $q \in \mathcal{Q}$ for σ that satisfies (42) and then forcing σ to "dwell" on q for at least some time τ_D before σ is allowed to switch to a new value. A dwell-time supervisor can then be defined as a hybrid system with a discrete state that is precisely the switching signal σ and a continuous state $\tau \in \mathbb{R}$ that is used as a timer to enforce the dwell-time. This hybrid system can be defined by the discrete transition equation

$$\sigma(t) = \begin{cases} \sigma^-(t) & \tau < \tau_D \text{ or } \sigma^-(t) \in \mathcal{S}[x(t)] \\ \min \mathcal{S}[x(t)] & \tau \geq \tau_D \text{ and } \sigma^-(t) \notin \mathcal{S}[x(t)] \end{cases}, \tag{43}$$

and the differential equation

$$\dot{\tau} = 1$$

that should hold almost everywhere. The continuous state τ is used for timing and should therefore be reset to zero whenever a switching occurs, i.e.,

$$\sigma(t) \neq \sigma^-(t) \quad \Rightarrow \quad \tau = 0,$$

This logic can perhaps be better understood by the diagram in Figure 3. In (43) and Figure 3 we assume that a partial order is defined in every $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ so that $\min \mathcal{S}[x(t)]$ is meaningful, but there is no reason to require the same partial order for all values of z. In fact, often one needs distinct partial orders for different values of z (cf. Section 3.2). In case $\mathcal{S}[z]$ is a continuum, suitable assumptions are also needed to make sure that the minimum does exists.

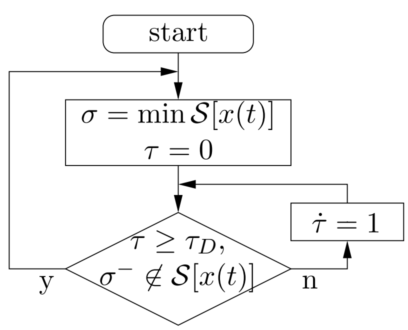

Figure 3: Dwell-time supervisor

By construction, this supervisor guarantees that

$$\sigma \in \mathcal{S}[\tau_D]$$

and also that (42) holds at every switching time $t \geq 0$. In case, the switching time $\bar{t}$ that follows t occurs after $t + \tau_D$, (42) also holds on the interval $[t + \tau_D, \bar{t})$. However, (42) may be violated in $(t, t + \tau_D)$.

3.2. Hysteresis-based Supervisors

Hysteresis-based supervisors are typically used to enforce state-dependent constraints at all times. These supervisors are similar to the dwell-time supervisors defined in Section 3.1, except that they do not enforce the dwell-time constraint and therefore do not have the continuous state τ used for timing. A hysteresis supervisor is then simply defined by the discrete transition equation

$$\sigma(t) = \begin{cases} \sigma^-(t) & \sigma^-(t) \in \mathcal{S}[x(t)] \\ \min \mathcal{S}[x(t)] & \sigma^-(t) \notin \mathcal{S}[x(t)] \end{cases}, \tag{44}$$

or, equivalently, by the diagram in Figure 4.

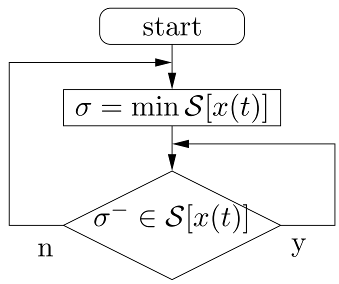

Figure 4: Hysteresis-based supervisor

By construction, this supervisor guarantees that on every interval $[0, T)$ on which the solution to (5), (44) exists, we have

$$\sigma(t) \in \mathcal{S}[x(t)], \qquad t \in [0, T).$$

With a hysteresis-based supervisor, "slow switching" can only be achieved by judiciously selecting the $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$. Actually, if these sets are not properly chosen, the solution to (5) and (44) may even fail to exist because of chattering, i.e., a solution that would require an infinite number of switchings in finite time. A typical example, is given by the switched system

$$\dot{x} = \sigma,$$

where $x \in \mathbb{R}$, $\sigma \in \mathcal{Q} := \{-1, 1\}$ and

$$\mathcal{S}[z] := \begin{cases} \{-1\} & z \geq 0 \\ \{+1\} & z < 0 \end{cases}, \qquad z \in \mathbb{R}.$$

In this case, the supervisor essentially implements a discontinuous control law and the differential equation does not have a solution in the sense of Carathéodory. A solution in the sense of Filippov (1964) exists in this case. However, this type of solutions are outside the scope of this chapter.

The chattering problem mentioned above can be avoided if we impose restrictions on the sets $\mathcal{S}[z] \subset \mathcal{Q}$, $z \in \mathbb{R}^n$. We proceed to derive some of these conditions. Consider the sets

$$\mathcal{X}_q := \{z \in \mathbb{R}^n : q \in \mathcal{S}[z]\}, \qquad q \in \mathcal{Q}. \tag{45}$$

As seen before, $\sigma \in \mathcal{S}[x]$ in the sense of (23) is equivalent to $x \in \mathcal{X}_\sigma$ in the sense of (24). A sufficient condition that excludes the possibility of chattering is that all the sets $\mathcal{X}_q$, $q \in \mathcal{Q}$ be open. This is because, if at some time $t \geq 0$, σ switched to q this means that

$$q \in \mathcal{S}[x(t)] \quad \Leftrightarrow \quad x(t) \in \mathcal{X}_q.$$

Since $\mathcal{X}_q$ is open, $x(t)$ is in the interior of this set and it will remain there for some time due to the continuity of $x(t)$. Therefore, q will remain inside $\mathcal{S}[x(t)]$ for some time and no switching will occur during this period. However, this still does not exclude the possibility of infinitely many switches on a finite interval. Indeed, it could happen that σ remains constant over intervals of smaller and smaller length, leading to a finite accumulation point $T < \infty$ for the switching times and to a solution that is only defined on $[0, T)$. Even though x remains uniformly bounded on $[0, T)$. To prevent this we need a stronger condition on the sets $\mathcal{X}_q$, $q \in \mathcal{Q}$.

We say that $\mathcal{X}_q$, $q \in \mathcal{Q}$ is a *uniformly open covering* of $\mathbb{R}^n$ if there exists a positive constant ϵ such that the ϵ-ball centered around every point $z \in \mathcal{X}_q$, $q \in \mathcal{Q}$ at the boundary of some $\mathcal{X}_{\bar{q}}$, $\bar{q} \in \mathcal{Q}$ is fully contained in $\mathcal{X}_q$. In this case, when σ switches to q at time t, σ will remain equal to q at least until x leaves the ϵ-ball centered around $x(t)$. Thus σ remains constant for an interval of time with length larger or equal to L/ϵ, where L is an upper-bound of the derivative of $\|x\|$. In case x is bounded (and therefore so is its derivative), we conclude that there must be a lower-bound on the difference between consecutive switching times. We can then use standard arguments to conclude that a solution to (5), (44) may not exist globally only due to finite escape, i.e., unboundedness of x in finite time. The following Lemma formalizes this observation:

Lemma 5. *Suppose that for every $q \in \mathcal{Q}$, the functions $A_q(x, d)$ are locally Lipschitz and the covering $\mathcal{X}_q$, $q \in \mathcal{Q}$ is uniformly open. If d is piecewise continuous and $[0, T)$ is the maximum interval over which the solution to* (5), (44) *exists, then there is only a finite number of switchings on every finite subset of* $[0, T)$ *and either $T = +\infty$ or x is unbounded on* $[0, T)$.

Requiring the $\mathcal{X}_q$ to be open is often too restrictive and would certainly exclude many of the state-dependent switchings considered in Section 2.3. An alternative condition that also avoids chattering can be formulated when all the sets $\mathcal{X}_q$ can be written either as

$$\mathcal{X}_q := \{z \in \mathbb{R}^n : W_q(z) \geq 0\} \tag{46}$$

or

$$\mathcal{X}_q := \{z \in \mathbb{R}^n : W_q(z) > 0\}, \tag{47}$$

for appropriately defined continuously differentiable functions $W_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$. In this case we can exclude the possibility of chattering by requiring that, when $x(t)$ is at the boundary of one of the $\mathcal{X}_q$—and therefore a switching to $q^* := \min \mathcal{S}[x(t)]$ may occur—either $x(t)$ is already in the interior of $\mathcal{X}_{q^*}$ or the differential equation $\dot{x} = A_{q^*}(x, d(t))$ moves x to the interior of this set. In either case, some time must elapse before further switching can occur. To achieve this, we can require that, for every $z \in \mathbb{R}^n$, $q \in \mathcal{Q}$,

$$W_q(z) = 0 \qquad \Rightarrow \qquad W_{q^*}(z) \geq \epsilon > 0 \qquad \text{or} \qquad \frac{\partial W_{q^*}}{\partial z}(z) A_{q^*}(z, d(t)) \geq \epsilon > 0, \quad \forall t \geq 0, \tag{48}$$

where $q^* := \min \mathcal{S}[z]$ and ϵ is a positive constant independent of z. This is because, when $x(t)$ is at the boundary of $\mathcal{X}_q$, $W_q\big(x(t)\big) = 0$ and therefore, we will either have $W_{q^*}\big(x(t)\big) \geq \epsilon > 0$ (which means that $x(t)$ is in the interior of $\mathcal{X}_{q^*}$) or $W_{q^*}\big(x(t)\big) = 0$ and $\frac{\mathrm{d}}{\mathrm{dt}}\big(W_{q^*}\big(x(t)\big)\big) \geq \epsilon > 0$ (which means that $x(t)$ will enter the interior of $\mathcal{X}_{q^*}$). The following can then be stated:

Lemma 6. *Suppose that, for every $q \in \mathcal{Q}$, the functions $A_q(x, d)$ are locally Lipschitz and the sets $\mathcal{X}_q$ are defined by* (46) *or by* (47), *with the W_q satisfying* (48). *If d is piecewise continuous and $[0, T)$ is the maximum interval over which the solution to* (5), (44) *exists, then then there is only a finite number of switchings on every finite subset of* $[0, T)$ *and either $T = +\infty$ or x is unbounded on* $[0, T)$.

Note that the sets $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ in (33) and also in (36) lead to $\mathcal{X}_q$, $q \in \mathcal{Q}$ of the form (46) and therefore this type of supervisor can be used to generate stabilizing switching signals, provided that the vector fields satisfy (48). Moreover, satisfying (48) often simply involves judiciously selecting the partial orders that define $q^* := \min \mathcal{S}[z]$.

The condition (48) could be relaxed by allowing the first k derivatives of $W_{q^*}\big(x(t)\big)$ to be zero, provided that its $(k+1)$th derivative is positive. This is still sufficient to guarantee that $x(t)$ will enter the interior of $\mathcal{X}_{q^*}$.

Lemma 6 can also be generalized to the case when the functions W_q are not continuously differentiable but are differentiable almost everywhere. In this case, when the derivative of W_{q^*} does not exists at z, the last inequality in (48) should be replaced by

$$\frac{\partial W_{q^*}}{\partial z}(\bar{z}) A_{q^*}(z, d(t)) \geq \epsilon > 0, \qquad \forall\, t \geq 0,$$

for $\bar{z}$ almost everywhere in some open neighborhood of z. Less conservative conditions are possible using concepts from nonsmooth analysis (cf., e.g., the standard textbooks by Aubin and Cellina, 1984; Clarke, 1989).

We consider next a specialized version of this supervisor for which it is actually possible to provide an upper bound on the number of switches that can occur on any finite interval. The supervisor in question was introduced by Hespanha (1998) and is called a *Scale-Independent Hysteresis Switching Logic*. This type of supervisor utilizes *monitoring signals* defined by

$$\mu_q(t) := \Pi\big(q, x(t)\big), \qquad t \geq 0,\ q \in \mathcal{Q}, \tag{49}$$

where $\Pi : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}$ is a *monitoring function* that is continuous with respect to the second argument, for frozen values of the first. The Scale-Independent Hysteresis Switching Logic attempts to keep σ equal to the index $q \in \mathcal{Q}$ of the smallest monitoring signal μ_q. However, to avoid chattering, it only switches to a different value in $\mathcal{Q}$ when the current monitoring signal grows above the smallest one by a pre-specified amount. This can be achieved by the discrete transition equation in (44) with

$$\mathcal{S}[z] := \{q \in \mathcal{Q} : \Pi(q, z) < (1+h)\Pi(\bar{q}, z),\ \forall \bar{q} \in \mathcal{Q}\}, \qquad z \in \mathbb{R}^n, \tag{50}$$

where h is a positive scalar called the *hysteresis constant*. This supervisory can also be defined by the diagram in Figure 5. To make sure that the sets $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ are nonempty we require that

$$\Pi(q, z) \geq \epsilon > 0, \qquad z \in \mathbb{R}^n, q \in \mathcal{Q}. \tag{51}$$

Although (51) only needs to hold for z that are reachable along solutions to (5), for simplicity we require this inequality to hold over the whole $\mathbb{R}^n$. Under this assumption, $\mathcal{S}[z]$ will always contain some q for which $\Pi(q, z)$ is sufficiently close to $\inf_{\bar{q} \in \mathcal{Q}} \Pi(\bar{q}, z)$. In the context of hybrid control, each μ_q typically provides a measure of the performance of the qth controller (cf. Figure 1).

From the definition of this logic, it is clear that the switching signal σ would not be affected if we replaced the monitoring signals μ_q, $q \in \mathcal{Q}$ by their scaled versions

$$\bar{\mu}_q(t) := \vartheta(t)\mu_q(t), \qquad q \in \mathcal{Q},\ t \geq 0,$$

where $\vartheta(t)$ is some positive function of time (cf. (50) and (51)). The term "scale-independence" arises precisely from this property. Scale-independence is useful because one can scale the

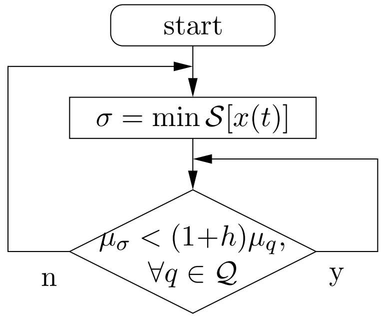

Figure 5: Scale-Independent Hysteresis Switching Logic

monitoring signals so that the $\bar{\mu}_q$, $q \in \mathcal{Q}$ have additional properties (such as, monotonicity) that are useful for analysis purposes (cf. below).

For this supervisor, the sets $\mathcal{X}_q$, $q \in \mathcal{Q}$ defined in (45) are given by

$$\mathcal{X}_q = \{z \in \mathbb{R}^n : \Pi(q, z) < (1 + h)\Pi(\bar{q}, z),\ \forall \bar{q} \in \mathcal{Q}\}, \qquad q \in \mathcal{Q}.$$

Since these sets are open, we conclude from Lemma 5 that chattering cannot occur and therefore the solution to (5), (44) may not exist globally only due to finite escape of the differential equation (5). Hespanha *et al.* (2000) went further and showed that it is actually possible to establish an upper bound on the number of switches that can occur in any finite interval, provided that all the monitoring signals are monotone nondecreasing:

Theorem 18 (Scale-Independent Hysteresis Switching). *Let $\mathcal{Q}$ be a finite set with m elements and assume that the solution $\{x, \sigma\}$ to (5), (44) is such that all monitoring signals $\mu_q := \Pi(q, x)$, $q \in \mathcal{Q}$ are monotone nondecreasing on the interval $[0, T)$ where the solution is defined. The switching signal σ is piecewise constant on $[0, T)$; for every $\ell \in \mathcal{Q}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1 + h)} \log\Big(\frac{\mu_\ell(t)}{\min_{q\in\mathcal{Q}} \mu_q(t_0)}\Big), \qquad 0 \leq t_0 < t < T; \tag{52}$$

and, when the monitoring signals are piecewise differentiable, we also have

$$\int_{t_0}^{t} \dot{\mu}_\sigma(\tau) d\tau \leq m\Big((1 + h)\mu_\ell(t) - \min_{q\in\mathcal{Q}} \mu_q(t_0)\Big), \qquad 0 \leq t_0 < t < T, \tag{53}$$

where $\dot{\mu}_\sigma(\tau)$ is defined to be equal to $\frac{\mathrm{d}\mu_q}{\mathrm{d}\tau}(\tau)$ on intervals where σ is constant and equal to $q \in \mathcal{Q}$.

When there is some μ_ℓ, $\ell \in \mathcal{Q}$ bounded on $[0, T)$, σ can only have a finite number of discontinuities on $[0, T)$ because of (52). This means that there must be a time $T^* < T$ beyond which σ is constant. Moreover, since $\sigma = \sigma(T^*)$ on $[T^*, T)$,

$$\mu_{\sigma(T^*)}(t) = \mu_{\sigma(T^*)}(T^*) + \mu_{\sigma(T^*)}(t) - \mu_{\sigma(T^*)}(T^*) \leq \mu_{\sigma(T^*)}(T^*) + \int_{T^*}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\mu_{\sigma(\tau)}(\tau)\Big) d\tau \quad t \in [T^*, T).$$

We therefore conclude that $\mu_{\sigma(T^*)}$ must be bounded on $[0, T)$ because of (53). Theorem 18 thus generalizes previous results by Hespanha (1998); Hespanha and Morse (1999a).

Sometimes the monitoring signals are not differentiable because they are generated by a monitoring function of the form

$$\Pi(q, z) := \min_{p\in\mathcal{P}_q} \tilde{\Pi}(p, z), \tag{54}$$

where $\{\mathcal{P}_q : q \in \mathcal{Q}\}$ is a parameterized family of compact sets and $\tilde{\Pi}$ a continuous function from $\mathcal{P} \times \mathbb{R}^n$ to $\mathbb{R}$, $\mathcal{P} := \cup_{q\in\mathcal{Q}}\mathcal{P}_q$. In this case the monitoring signals can be written as

$$\mu_q(t) := \min_{p\in\mathcal{P}_q} \tilde{\mu}_p(t), \qquad t \geq 0,\ q \in \mathcal{Q}, \tag{55}$$

with

$$\tilde{\mu}_p(t) := \tilde{\Pi}\big(p, x(t)\big), \qquad t \geq 0,\ p \in \mathcal{P}. \tag{56}$$

These monitoring signals arise in hybrid adaptive control and typically each $q \in \mathcal{Q}$ corresponds to a particular candidate controller and each $p \in \mathcal{P}_q$ to a particular process model that can be stabilized by that controller. Because of this the $\tilde{\mu}_p$ are called *process monitoring signals*, whereas the μ_q are called *control monitoring signals*. A similar convention is used to name the monitoring functions. In general, the minimization of p over $\mathcal{P}_q$ in (54) corresponds to finding the process model that best matches the measured data, among those stabilized by the qth controller (for details see, e.g., Liberzon *et al.*, 2000).

We say that a piecewise constant signal ρ taking values in $\mathcal{P}$ is $\{\mathcal{P}_q : q \in \mathcal{Q}\}$-consistent with a switching signal σ on an interval (τ, t) when

1. $\rho(s) \in \mathcal{P}_{\sigma(s)}$ for all $s \in (\tau, t)$,
2. the set of discontinuities of ρ on (τ, t) is a subset of the set of discontinuities of σ.

In hybrid adaptive control, when a signal ρ corresponding to a sequence of process models is $\{\mathcal{P}_q : q \in \mathcal{Q}\}$-consistent with a signal σ corresponding to a sequence of candidate controllers, we have that the candidate controller indexed by $\sigma(s)$ stabilizes the process model $\rho(s)$, for every frozen time s.

Although, in general, the control monitoring signals defined by (55) are not differentiable, the process monitoring signals in (56) are. It is therefore useful to consider a corollary of Theorem 18 in terms of process monitoring signals:

Corollary 19. *Let $\mathcal{Q}$ be a finite set with m elements and assume that the solution $\{x, \sigma\}$ to (5), (44) is such that all process monitoring signals $\tilde{\mu}_p := \tilde{\Pi}(p, x)$, $p \in \mathcal{P}$ are monotone nondecreasing on the interval $[0, T)$ where the solution is defined. The switching signal σ is piecewise constant on $[0, T)$; for every $\ell \in \mathcal{Q}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\tilde{\mu}_\ell(t)}{\min_{p\in\mathcal{P}} \tilde{\mu}_p(t_0)}\right), \qquad 0 \leq t_0 < t < T; \tag{57}$$

and, when the process monitoring signals are differentiable, we also have that there exists a signal ρ, which is $\{\mathcal{P}_q : q \in \mathcal{Q}\}$-consistent with σ on (t_0, t), such that

$$\int_{t_0}^{t} \dot{\tilde{\mu}}_\rho(\tau) d\tau \leq m\Big((1+h)\tilde{\mu}_\ell(t) - \min_{p\in\mathcal{P}} \tilde{\mu}_p(t_0)\Big), \qquad 0 \leq t_0 < t < T, \tag{58}$$

where $\dot{\tilde{\mu}}_\rho(\tau)$ is defined to be equal to $\frac{\mathrm{d}\tilde{\mu}_q}{\mathrm{d}\tau}(\tau)$ on intervals where ρ is constant and equal to $p \in \mathcal{P}$.

4. Case Studies

We conclude this chapter by presenting two case studies that illustrate how some of the tools presented in the previous sections can be used to analyze and design hybrid controllers.

4.1. Vision-based Control of a Flexible Manipulator

Consider the system in Figure 6(a) consisting of a single-link flexible manipulator driven by a field-controlled DC Motor. Assuming small bending of the flexible link and considering only the dominate flexible mode, the dynamics of the manipulator can be approximated by the four-dimensional system in Figure 6(b). This system can be modeled by

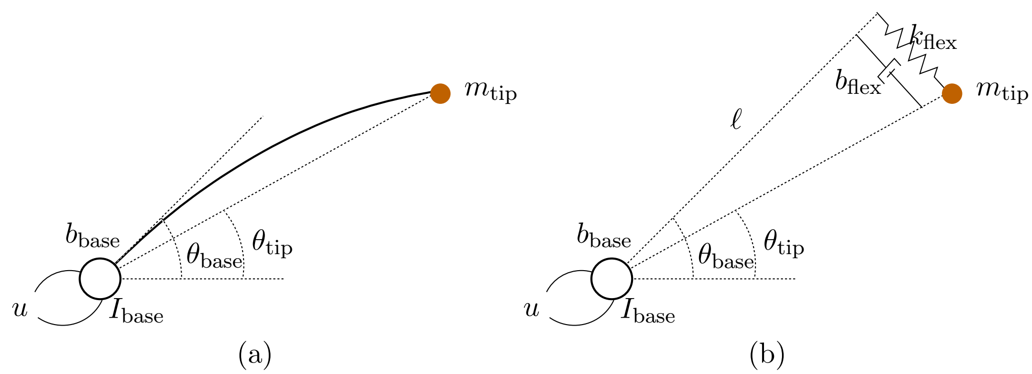

Figure 6: Flexible Manipulator: (a) infinite dimensional system, (b) finite dimensional approximation.

$$m_{\text{tip}}\ell^2\ddot{\theta}_{\text{tip}} = \ell k_{\text{flex}}(\theta_{\text{base}} - \theta_{\text{tip}}) + \ell b_{\text{flex}}(\dot{\theta}_{\text{base}} - \dot{\theta}_{\text{tip}}), \tag{59}$$

$$I_{\text{base}}\ddot{\theta}_{\text{base}} = -b_{\text{base}}\dot{\theta}_{\text{base}} + \ell k_{\text{flex}}(\theta_{\text{tip}} - \theta_{\text{base}}) + \ell b_{\text{flex}}(\dot{\theta}_{\text{tip}} - \dot{\theta}_{\text{base}}) + k_{\text{motor}}u, \tag{60}$$

where ℓ denotes the length of the link, $k_{\text{flex}}, b_{\text{flex}}$ the parameters of the dominant flexible mode, m_{tip} the mass at the tip, I_{base} the base moment of inertia, and k_{motor} the motor gain. The following numerical values will be used here:

$$\frac{k_{\text{flex}}}{\ell m_{\text{tip}}} = 2.0, \qquad \frac{b_{\text{flex}}}{\ell m_{\text{tip}}} = 0.1, \qquad \frac{b_{\text{base}}}{I_{\text{base}}} = .05, \qquad \frac{m_{\text{tip}}\,\ell^2}{I_{\text{base}}} = .01, \qquad \frac{k_{\text{motor}}}{I_{\text{base}}} = 1.$$

The control objective is to asymptotically stabilize the origin of the system using feedback from measurements of θ_{base} and θ_{tip}. The signal θ_{base} is simple to measure using an encoder at the base, but θ_{tip} provides some challenge. We assume here that θ_{tip} is measured using a camera that observes a neighborhood of the target position $\theta_{\text{tip}} = 0$. Because of its limited field of view, we only obtain information from the camera when $|\theta_{\text{tip}}| \leq \theta_{\max}$. The output of the process is then

$$y := \begin{cases} \begin{bmatrix} \theta_{\text{base}} & \theta_{\text{tip}} \end{bmatrix}^T & |\theta_{\text{tip}}| \leq \theta_{\max} \\ \theta_{\text{base}} & |\theta_{\text{tip}}| > \theta_{\max}. \end{cases} \tag{61}$$

Here, we will take $\theta_{\max} = 1$. Although this system is observable through θ_{base}, when the base inertia I_{base} is much larger than $m_{\text{tip}}\,\ell^2$, the observability matrix that corresponds to the single output θ_{base} becomes almost singular and changing the poles corresponding to the flexible mode makes the closes-loop system very sensitive to noise and unmodeled dynamics. A natural solution to this problem consists of designing two controllers and switch among them. The first controller operates when $|\theta_{\text{tip}}| > \theta_{\max}$ and therefore no position information for the tip is available. For the reasons mentioned above, this controller will not attempt to move the poles corresponding to the flexible mode and the closed-loop will exhibit two under-damped complex conjugate poles. The second controller will operate when $|\theta_{\text{tip}}| \leq \theta_{\max}$. In this region of operation, the tip position is available and therefore it is possible to avoid the lightly damped poles. We designed both controllers using LQG so as to minimize

$$\lim_{T\to\infty} \mathrm{E}\left[\frac{1}{T}\int_0^T \left(\theta_{\text{tip}}^2 + \dot\theta_{\text{tip}}^2 + 10^{-6}u^2\right)dt\right],$$

when the state and output equations are corrupted by additive white noise d and n, respectively, with

$$\mathrm{E}[dd^T] = I, \qquad \mathrm{E}[nn^T] = I, \qquad \mathrm{E}[nd^T] = 0.$$

The resulting closed-loop system falls under the hybrid control architecture shown in Figure 1. The feedback system consisting of the process (59)–(61) and the multi-controller that effectively switches between the first and second controllers can be written as the following eight-dimensional system

$$\dot x = A_\sigma x, \tag{62}$$

where $x := \begin{bmatrix}\theta_{\text{tip}} & \dot\theta_{\text{tip}} & \theta_{\text{base}} & \dot\theta_{\text{base}} & x_C^T\end{bmatrix}^T$, x_C is the state of the multi-controller, σ is the switching signal taking values on $\{1,2\}$, and A_1 and A_2 are the matrices that define the closed-loop systems with the first and second controllers, respectively. The requirement placed on the supervisor is that the first controller be used only when $|\theta_{\text{tip}}| \le \theta_{\max}$. If this condition fails, the second controller must be used.

It turns out that there is no common quadratic Lyapunov function for the systems

$$\dot z = A_q z, \qquad q \in \{1,2\}.$$

This was verified numerically, by showing that there is no feasible solution P to (16). We therefore opted for a stability argument using multiple Lyapunov functions. To this effect we considered the following covering of the state-space: $\mathcal{X}_1$ denotes the subset of the state-space for which $|\theta_{\text{tip}}| \le \theta_{\max}$, $\mathcal{X}_2$ the subset for which $\theta_{\text{tip}} \ge \theta_{\max}$, and $\mathcal{X}_3$ the subset for which $\theta_{\text{tip}} \le -\theta_{\max}$. The first controller should only be used in $\mathcal{X}_1$, whereas the second should be used either in $\mathcal{X}_2$ or $\mathcal{X}_3$. For now we do not specify which controller to use over the boundary between the $\mathcal{X}_q$. For consistency, we define $A_3 := A_2$ and assume that the switching signal σ actually takes values in the set $\mathcal{Q} := \{1,2,3\}$. The relevant state-dependent family of switching signals can then be defined by

$$\mathcal{S}[z] = \{q \in \mathcal{Q} : E_q \begin{bmatrix} z^T & 1\end{bmatrix}^T \succeq 0\}, \qquad z \in \mathbb{R}^8,$$

and

$$E_1 := \begin{bmatrix} -c_{\text{tip}} & \theta_{\max} \\ c_{\text{tip}} & \theta_{\max}\end{bmatrix}, \qquad E_2 := \begin{bmatrix} c_{\text{tip}} & -\theta_{\max}\end{bmatrix}, \qquad E_3 := \begin{bmatrix} -c_{\text{tip}} & -\theta_{\max}\end{bmatrix},$$

where $c_{\text{tip}} := \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$. In this case, $\bar{\mathcal{S}}[z] = \mathcal{S}[z]$ for every $z \in \mathbb{R}^8$. The boundary between $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$, $(q_1, q_1) \in \partial\mathcal{Q} := \{(1,2), (1,3)\}$ is defined by

$$f_{q_1 q_2}^T \begin{bmatrix} z^T & 1\end{bmatrix}^T = 0,$$

where

$$f_{1\,2} := \begin{bmatrix} c_{\text{tip}} & -\theta_{\max}\end{bmatrix}, \qquad f_{1\,3} := \begin{bmatrix} c_{\text{tip}} & \theta_{\max}\end{bmatrix}.$$

It turns out that one can find vectors k_{12}, k_{13} and symmetric matrices P_q, $q \in \mathcal{Q}$ for which (38) and (39) hold with $\epsilon = \delta = 10^{-3}$ and $U_q = W_q = 0$, $q \in \mathcal{Q}$. We found these vectors and matrices using MATLAB's LMI Toolbox. Because of Lemma 4, we then conclude that $\{V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z^T & 1 \end{bmatrix}^T : q \in \mathcal{Q}\}$ is a family of multiple Lyapunov functions for $\mathcal{S}[x]$, for which the matching condition (31) holds. This means that (62) is uniformly asymptotically stable over $\mathcal{S}[x]$.

To generate the switching signal σ, we used the hysteresis-based supervisor in Figure 4. To specify this supervisor it remains to define the partial orders used to select the value of the switching signal at switching times. We will select these so as to be able to apply Lemma 6. To this effect, note that we can write the sets $\mathcal{X}_q$, $q \in \mathcal{Q}$ defined in (45) as

$$\mathcal{X}_q = \{z \in \mathbb{R}^n : W_q(z) \geq 0\}, \qquad\qquad q \in \mathcal{Q},$$

where

$$W_1(z) := \min\{-E_2 \begin{bmatrix} z^T & 1 \end{bmatrix}^T, -E_3 \begin{bmatrix} z^T & 1 \end{bmatrix}^T\}, \;\; W_2(z) := E_2 \begin{bmatrix} z^T & 1 \end{bmatrix}^T, \;\; W_3(z) := E_3 \begin{bmatrix} z^T & 1 \end{bmatrix}^T.$$

It is then straightforward to show that the hypothesis of Lemma 6 hold when we select the partial orders for the $\mathcal{S}[z]$, $z \in \mathbb{R}^n$ so that (i) when $c_{\text{tip}} z = \theta_{\max}$,

$$q^* := \min \mathcal{S}[z] = \begin{cases} 1 & c_{\text{tip}} A_1 z < 0 \\ 2 & c_{\text{tip}} A_2 z > 0 \\ 1 & c_{\text{tip}} A_1^k z = c_{\text{tip}} A_2^k z = 0, \; c_{\text{tip}} A_1^{k+1} z < 0, \; \text{for some } k \geq 1 \\ 2 & c_{\text{tip}} A_1^k z = c_{\text{tip}} A_2^k z = 0, \; c_{\text{tip}} A_1^{k+1} z \geq 0, \; c_{\text{tip}} A_2^{k+1} z > 0, \; \text{for some } k \geq 1 \end{cases}$$

and (ii) when $c_{\text{tip}} z = -\theta_{\max}$,

$$q^* := \min \mathcal{S}[z] = \begin{cases} 1 & c_{\text{tip}} A_1 z > 0 \\ 3 & c_{\text{tip}} A_3 z < 0 \\ 1 & c_{\text{tip}} A_1^k z = c_{\text{tip}} A_3^k z = 0, \; c_{\text{tip}} A_1^{k+1} z > 0, \; \text{for some } k \geq 1 \\ 3 & c_{\text{tip}} A_1^k z = c_{\text{tip}} A_3^k z = 0, \; c_{\text{tip}} A_1^{k+1} z \leq 0, \; c_{\text{tip}} A_3^{k+1} z < 0, \; \text{for some } k \geq 1 \end{cases}$$

Actually, we need here the relaxed version of the condition (48) that allows the first k derivatives of $W_{q^*}\big(x(t)\big)$ to be zero, provided that its $(k+1)$th derivative is positive. Figure 7 shows a simulation of the closed-loop system. The plots (a) on the left show the response when only the controller that does not utilize the θ_{tip} is used, i.e., σ is fixed and equal to 2 in (62). The plots (b) on the right show the response of the hybrid controller with the hysteresis-based supervisor defined above. A dramatic reduction in the settling time was achieved with the hybrid controller.

4.2. Hybrid Adaptive Set-point Control

The problem addressed here is the set-point control of an imprecisely modeled process. In particular, we want to generate the control input u to the process so as to drive its output y to a constant reference r. The process has two other exogenous inputs that cannot be measured: a bounded measurement noise signal $\mathbf{n}$ and a bounded disturbance $\mathbf{d}$. For simplicity the signals u, y, $\mathbf{n}$, and $\mathbf{d}$ are scalar. The process is assumed linear, time-invariant, with a stabilizable (through u) and detectable realization

$$\dot{x}_P = A_P x + B_P u + D_P \mathbf{d}, \qquad\qquad y = C_P x + \mathbf{n}, \tag{63}$$

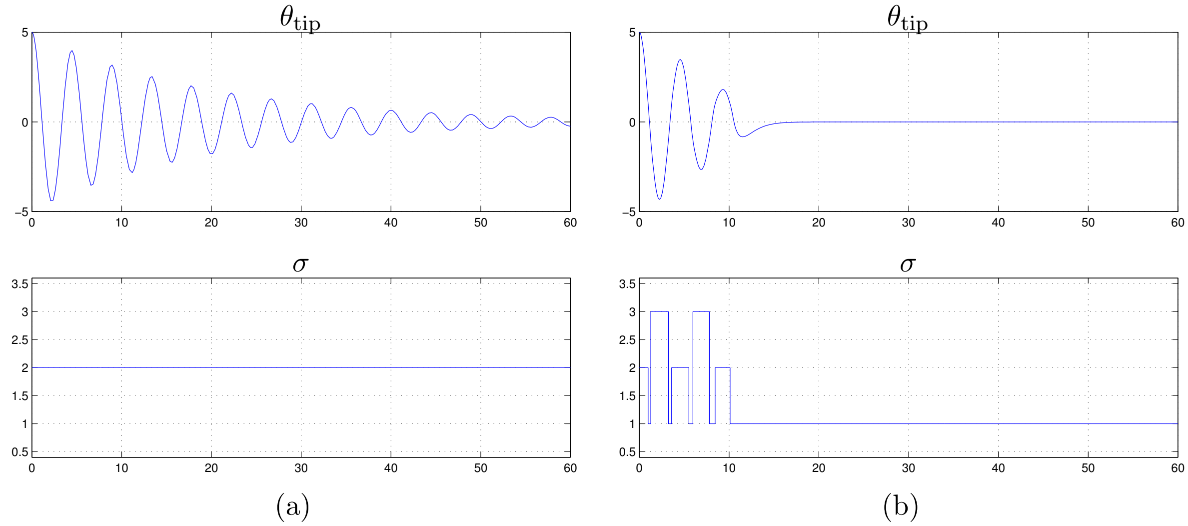

Figure 7: Simulation of the flexible manipulator: (a) fixed controller, (b) hybrid controller.

but precise values for A_P, B_P, C_P, D_P are not known. It is known, however, that the process' transfer function τ, from u to y, belongs to a family of admissible transfer functions $\mathcal{N} := \{\nu_q : q \in \mathcal{Q}\}$ where q is an unknown parameter taking values in some parameter set $\mathcal{Q}$.

The solution proposed by Morse (1996, 1997) to solve this problem is based on Certainty Equivalence and starts with the selection of a family of linear, time-invariant *candidate controllers* $\mathcal{C} := \{\kappa_q : q \in \mathcal{Q}\}$. Each κ_q would make the closed-loop system in Figure 8 asymptotically stable if the process transfer function τ was equal to ν_q. To avoid pole-zero cancellations it is assumed that τ does not have transmission zeros at the origin.

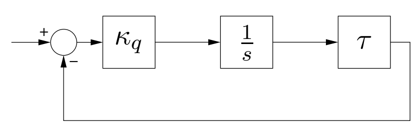

Figure 8: Feedback configuration.

In case we knew that the actual process transfer function τ was equal to a specific $\nu_q \in \mathcal{N}$, stability of the closed loop could be achieved with the nonadaptive, linear, time-invariant controller κ_q. Since the process transfer function is not known in advance we build a multi-controller that effectively allows switching between the controllers in $\mathcal{C}$. The multi-controller can be defined by

$$\dot{x}_C = F_\sigma x_C + G_\sigma \mathbf{e_T}, \qquad v = H_\sigma x_C + J_\sigma \mathbf{e_T}, \qquad \dot{u} = v, \qquad \mathbf{e_T} = r - y, \tag{64}$$

where each (F_q, G_q, H_q, J_q) is a n_C-dimensional stabilizable and detectable realization of κ_q and $\sigma : [0, \infty) \to \mathcal{Q}$ is a switching signal that determines which candidate controller is placed into the feedback loop. Here, we are interested in estimator-based supervisors to generate σ. An estimator-based supervisor consists of three blocks to be described shortly: a multi-estimator, a monitoring signal generator, and a switching logic. The overall hybrid control system is shown in Figure 9.

The *multi-estimator* is a linear, time-invariant system whose inputs are the outputs of the process and multi-controller and whose outputs are the *output estimation errors* e_q, $q \in \mathcal{Q}$. Each e_q is a signal that would converge to zero if the process transfer function τ was equal to

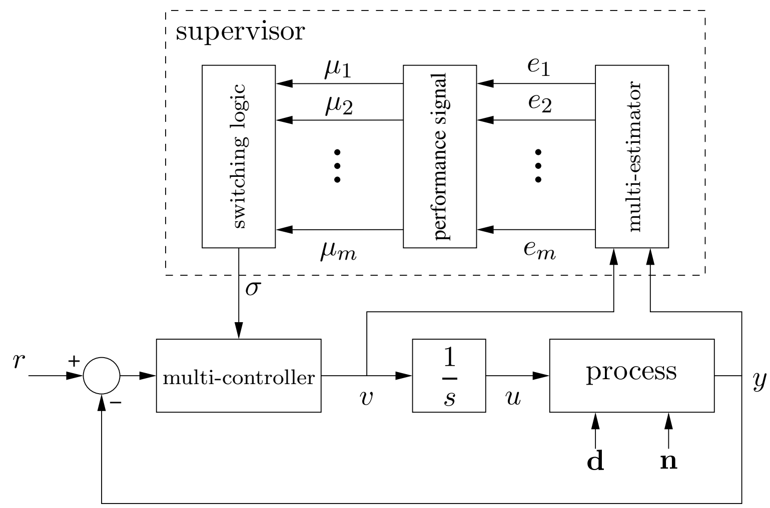

Figure 9: Estimator-based hybrid adaptive control architecture.

the transfer function ν_q. The reader is referred to Morse (1996, 1997) for the precise structure of the multi-estimator. Denoting by x the combined state of the multi-estimator and multi-controller (excluding the integrator), the evolution of x is determined by

$$\dot{x} = A_\sigma x + d_\sigma e_\sigma, \tag{65}$$

$$\mathbf{e_T} = c_{p^*} x + e_{p^*}, \tag{66}$$

where A_q, d_q, c_q, $q \in \mathcal{Q}$ are appropriately defined matrices, and q^* is the element of $\mathcal{Q}$ for which $\tau = \nu_{q^*}$. Equation (65) is obtained from equation (23) in (Morse, 1996) with $l = \sigma$ and (66) is obtained from equation (26) in (Morse, 1996) with $l = q^*$. It is also known that there exists a positive constant λ_0 for which every $\lambda_0 I + A_q$ is asymptotically stable (cf. Remark 4 in (Morse, 1996)). Moreover, from equation (28) in (Morse, 1996), one concludes that e_{q^*} is bounded and

$$\int_0^t e^{2\lambda\tau} e_{q^*}(\tau)^2 d\tau \leq c_n e^{2\lambda t} + c_0, \qquad t \geq 0, \tag{67}$$

$$\|e_{q^*}(t)\| \leq d_n + d_0 e^{-\lambda t}, \qquad t \geq 0, \tag{68}$$

where λ can be any constant in $(0, \lambda_0)$, c_0, d_0 are positive constants that depends only on initial conditions, and c_n, d_n are positive constants that depends only on upper bounds on the norms of **n** and **d**.

The *monitoring signal generator* takes as inputs the output estimation errors e_q, $q \in \mathcal{Q}$, and produces the *monitoring signals* μ_q, $q \in \mathcal{Q}$ defined by

$$\dot{\tilde{\mu}}_q = -2\lambda\tilde{\mu}_q + e_q^2, \qquad \mu_q = \tilde{\mu}_q + \epsilon_\mu, \qquad q \in \mathcal{Q}, \tag{69}$$

with $\lambda \in (0, \lambda_0)$ and $\epsilon_\mu > 0$ constant. The system (69) is initialized so that $\tilde{\mu}_q(0) \geq 0$, $q \in \mathcal{Q}$. Essentially, each monitoring signal μ_q provides a measure of the size of the corresponding error e_q, with a forgetting factor defined by λ.

The *switching logic* generates the switching signal σ based on the values of the monitoring signals μ_q, $q \in \mathcal{Q}$. The logic used here is the Scale-Independent Hysteresis Switching Logic defined in Section 3.2.

Suppose now that we define *scaled monitoring signals* $\bar{\mu}_q := \vartheta\mu_q$, with $\vartheta(t) := e^{2\lambda t}$, $t \geq 0$. From (69) one concludes that, for every $t \geq t_0 \geq 0$,

$$\bar{\mu}_q(t) = \tilde{\mu}_q(t_0) + e^{2\lambda t}\epsilon_\mu + \int_{t_0}^t e^{2\lambda\tau} e_q(\tau)^2 d\tau, \tag{70}$$

and therefore each $\bar{\mu}_q$ is always monotone increasing and never smaller than ϵ_μ. By the Scale-Independent Hysteresis Switching Theorem 18, we can then conclude that, for every $\ell \in \mathcal{Q}$ and $0 \le t_0 \le t < T$,

$$N_\sigma(t, t_0) \le 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\mu}_\ell(t)}{\inf_{q \in \mathcal{Q}} \bar{\mu}_q(t_0)}\right) \tag{71}$$

and

$$\int_{t_0}^{t} \left(2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau} e_\sigma^2\right) d\tau \le m\Big((1+h)\bar{\mu}_\ell(t) - \inf_{q \in \mathcal{Q}} \bar{\mu}_q(t_0)\Big). \tag{72}$$

In (72) we used the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\bar{\mu}_{\sigma(\tau)}(\tau)\big) = 2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau}e_\sigma^2$, wherever the derivative exists. Now, from (67) and (70) we obtain $\bar{\mu}_{q^*}(t) \le e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0$, $t \ge 0$, where $\bar{c}_0 := c_0 + \tilde{\mu}_{q^*}(0)$. From this, (71)–(72) with $\ell := q^*$, and the fact that $\bar{\mu}_q(t_0) \ge e^{2\lambda t_0}\epsilon_\mu$, $q \in \mathcal{Q}$, we conclude that

$$N_\sigma(t, t_0) \le 1 + m + \frac{m \log\left(e^{2\lambda(t-t_0)}\Big(1 + \frac{c_n}{\epsilon_\mu}\Big) + \frac{\bar{c}_0}{\epsilon_\mu}\right)}{\log(1+h)},$$

$$\int_{t_0}^{t} e^{2\lambda\tau} e_\sigma^2 d\tau \le m\Big((1+h)\left(e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0\right) - e^{2\lambda t_0}\epsilon_\mu\Big) - \left(e^{2\lambda t} - e^{2\lambda t_0}\right)\epsilon_\mu,$$

for every $t \ge t_0 \ge 0$. Since for $a, b > 0$, $\log(a+b) \le \log(2a) + \log(2b)$, we also conclude that

$$N_\sigma(t, t_0) \le N_0 + \frac{t - t_0}{\bar{\tau}_D}, \tag{73}$$

$$\int_{t_0}^{t} e^{2\lambda\tau} e_\sigma^2 d\tau \le \bar{m} e^{2\lambda t} c_n + \bar{m}\bar{c}_0 - (m-1)e^{2\lambda t_0}\epsilon_\mu + (\bar{m}-1)e^{2\lambda t}\epsilon_\mu, \tag{74}$$

with $\tau_D := \frac{\log(1+h)}{2\lambda m}$, $\bar{m} = m(1+h)$, and

$$N_0 := 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{4\bar{c}_0}{\epsilon_\mu}\Big(1 + \frac{c_n}{\epsilon_\mu}\Big)\right).$$

Now, because of Lemma 1 and Theorem 13, there is a finite constant τ_D^* such that (65) has input-to-state $e^{\lambda t}$-weighted L_2-to-L_∞ norm uniformly bounded over $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. If we then choose λ and h so that $\frac{\lambda}{\log(1+h)} \le \gamma := \frac{1}{2m\tau_D^*}$, we get $\tau_D \ge \tau_D^*$ and the output σ of the switching logic is guaranteed to be in $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. From this and (74) one concludes that x is bounded and, because of (64) and (66), $\mathbf{e_T}$ and v are also bounded. The boundedness of u and the internal state of the process follows from the detectability of the cascade formed by the integrator in (64) and the process (63). The following can then be stated.

Theorem 20. *There exists a positive constant γ such that, whenever $\frac{\lambda}{\log(1+h)} \le \gamma$, all signals remain bounded, for every bounded* $\mathbf{n}$ *and* $\mathbf{d}$*, and every initialization of the close-loop system with with $\tilde{\mu}_q(0) \ge 0$, $q \in \mathcal{Q}$.*

The previous analysis assumed that the family $\mathcal{N} := \{\nu_q : q \in \mathcal{Q}\}$ of admissible transfer functions for the process was finite. Liberzon *et al.* (2000) extended this type of analysis to the case where $\mathcal{N}$ has infinitely many elements that may even exhibit unmodeled dynamics. This extension requires Corollary 19.

Appendix

Lemma 7. *Given a matrix $F \in \mathbb{R}^{n\times m}$ whose columns are orthonormal (i.e., $F^T F = I_m$) and a symmetric matrix $P \in \mathbb{R}^n$ the condition*

$$F^T z = 0 \quad \Rightarrow \quad z^T P z = 0, \qquad\qquad \forall z \in \mathbb{R}^n, \tag{75}$$

is equivalent to the existence of a matrix $G \in \mathbb{R}^{n\times m}$ such that

$$P = GF^T + FG^T. \tag{76}$$

Before proceeding note that the requirement that $F^T F = I_m$ could be relaxed to $F^T F = cI_m$ for some scalar c. This holds trivially when F is a column vector ($m = 1$).

Proof of Lemma 7. Equation (76) clearly implies (75), as for the converse we start by noting that $F^T(I_n - FF^T)x = 0$, for every $x \in \mathbb{R}^n$. This is because because $F^T F = I$. We must then have

$$x^T(I_n - FF^T)P(I_n - FF^T)x = 0, \qquad\qquad \forall x \in \mathbb{R}^n.$$

This means that $(I_n - FF^T)P(I_n - FF^T)$ must be skew symmetric. But since this matrix is also symmetric, we must have $(I_n - FF^T)P(I_n - FF^T) = 0$. Expanding this product, we conclude that (76) holds with $G := PF^T - \frac{1}{2}FF^T PF$. ■

Glossary

Hybrid system: Dynamical system that combines continuous with discrete-valued variables.

Supervisory control: Hierarchical control structure in which a high-level logic orchestrates the switching among several low-level controllers.

Finite escape time: Is said to occur when the solution to a differential equation becomes unbounded in finite time.

Chattering or Zeno phenomenon: Is said to occur when the solution to a hybrid system exhibits infinitely many discontinuities in a finite interval of time.

Class $\mathcal{K}$: Set of all continuous functions $\alpha : [0,\infty) \to [0,\infty)$ that are zero at zero, strictly increasing, and continuous.

Class $\mathcal{K}_\infty$: Subset of $\mathcal{K}$ consisting of those functions that are unbounded.

Class $\mathcal{KL}$: Set of continuous functions $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ which, for every fixed value of the second argument, are of class $\mathcal{K}$ when regarded as functions of the first argument, and that have $\lim_{\tau\to\infty} \beta(s,\tau) = 0$ for every fixed $s \geq 0$.

Realization of a transfer function: Quadruple of matrices (A, B, C, D) such that $C(sI - A)^{-1}B + D$ equals the transfer function.

Dwell-time: Minimum time interval between consecutive discontinuities of a piecewise constant signal.

Covering of a set: Family of sets (not necessarily disjoint) whose union contains the set.

Bibliographical Sketch

João P. Hespanha was born in Coimbra, Portugal, in 1968. He received the Licenciatura and the M.S. degree in electrical and computer engineering from Instituto Superior Técnico, Lisbon, Portugal, in 1991 and 1993, respectively, and the M.S. and Ph.D. degrees in electrical engineering and applied science from Yale University, New Haven, Connecticut, in 1994 and 1998, respectively. For his PhD work, Dr. Hespanha received Yale University's Henry Prentiss Becton Graduate Prize for exceptional achievement in research in Engineering and Applied Science.

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