Robust Stability under Asynchronous Sensing and Control

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Abstract— **We study robust stability analysis for networked control systems in which a sensor and an output-feedback controller operate asynchronously. This clock asynchronism leads to the uncertainty of sampling instants. Moreover, we assume the polytopic uncertainty of the plant. We first transform the closed-loop system into an impulsive system, by considering an extended variable that consists of the states of the continuous-time plant and the discrete-time controller. Next, using a continuous-time Lyapunov functional that incorporates the discrete-time state of the digital controller, we give a sufficient condition for robust stability of the closed-loop system in terms of linear matrix inequalities. Finally, we illustrate the obtained result with numerical simulations.**

I. Introduction

Uncertainty in the space domain such as parameter perturbation and disturbances/noises has been extensively studied in the robust control theory; see, e.g., [1], [2] and many references therein. On the other hand, control systems have uncertainty in the time domain as well [3], [4], but relatively little work has been done on time-domain uncertainty. Our goal is to analyze how large uncertainty in both the space and time domains can be allowed without compromising the closed-loop stability.

In networked control systems, one of the major sources of time-domain uncertainties is a synchronization error between the local subsystems. As surveyed in [5], [6], many synchronization algorithms have been developed, and easy access to global clocks such as GPS and radio clocks leads to high-precision synchronization in practical situations. However, there is a fundamental limitation on clock synchronization due to variable delays [7]. Furthermore, the signals of GPS and radio clocks are not ubiquitously available, and it is reported in [8] that the GPS-based synchronization is vulnerable against adversarial attacks.

Asynchronous dynamical systems have been investigated in various fields including engineering and biology. An observer-based control has been proposed for networked controlled systems under synchronization errors and parametric uncertainty in [9]. For systems with asynchronous sensing and control, stability analysis [10], L^2-gain analysis [11], and limitations on the clock offset tolerable for stabilization [12], [13] have been studied. The time measure of the optimal controller in [14] is a stochastic process subject to noise. The authors in [15] have compensated clock offsets and skews for the timestamp-based synchronization of multiple plants over networks. The experimental results in [16], [17] have indicated that human subjects potentially learn temporal uncertainty.

In this paper, we study robust stability analysis for systems that have variable clock offsets between the sensor and the digital controller. We assume that the system has polytopic uncertainty (in the space domain) as, e.g., in [18], [19]. We give a sufficient condition for the closed-loop stability via linear matrix inequalities (LMIs). The proposed method is illustrated with a numerical simulation by showing how large space/time-domain uncertainty would be allowed by a given controller.

A major difficulty in the analysis of the closed-loop system stems from the fact that the system has continuous-time and discrete-time state variables. In [9], a continuous-time controller is used with the input-delay approach [20]–[22], and hence the closed-loop system have only continuous-time states. Although the authors of [10] avoid this difficulty by discretizing the closed-loop system, there is a drawback that the parameters of the discretized system necessarily depends on those of the original plant in a nonlinear way, which brings negligible conservativeness for stability analysis.

Instead, we represent the closed-loop system as an impulsive system as done in [23]–[28] for systems with variable/constant delays and aperiodic sampling. This representation has the parameters depending linearly on those of the original system, and therefore allows us to more efficiently address the polytopic uncertainty in the original system. Moreover, we describe the state of the digital controller by a piecewise constant function and construct a Lyapunov functional that incorporates both continuous-time and discrete-time states.

This paper is organized as follows. In Section II, we introduce the closed-loop system and basic assumptions, and then formulate our problem. Section III is devoted to the main result, and we provide its proof in Section IV. In Section V, we discuss the advantage and the disadvantage of the discretization of the closed-loop system. We illustrate the proposed method with a numerical simulation in Section VI, and concluding remarks are given in Section VII.

Notation and definitions: For a real matrix M, let $M^\top$ be its transpose. For a real square matrix Q, define $\mathrm{He}(Q) := Q + Q^\top$. We denote the Euclidean norm of a real vector v by $\|v\| := (v^\top v)^{1/2}$.

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For a piecewise continuous function ϕ, let us denote its left-sided limit at time t by

$$\phi(t^-) := \lim_{h \searrow 0} \phi(t-h)$$

The upper right-hand derivative of ϕ with respect to time t is denoted by $\dot{\phi}$, that is,

$$\dot{\phi}(t) := \lim_{h \searrow 0} \frac{\phi(t+h) - \phi(t)}{h}.$$

Let $\mathcal{W}_h$ denote the space of functions $\phi: [-h, 0] \to \mathbb{R}^n$ that are absolutely continuous in $[-h, 0)$ and have the square integrable first-order derivatives in $[-h, 0]$. The norm of $\mathcal{W}_h$ is defined by

$$\|\phi\|_{\mathcal{W}_h} := \max_{\theta \in [-h,0]} \|\phi(\theta)\| + \left(\int_{-h}^{0} \left\| \dot{\phi}(s) \right\|^2 ds \right)^{1/2}.$$

We denote by $\mathcal{U}_h$ the space of functions $\psi: [-h, 0] \to \mathbb{R}^n$ that are piecewise continuous in $[-h, 0)$. The norm of $\mathcal{U}_h$ is defined by

$$\|\psi\|_{\mathcal{U}_h} := \max_{\theta \in [-h,0]} \|\psi(\theta)\|.$$

For two normed linear spaces W and U, we define the direct sum $W \oplus U$ by

$$W \oplus U := \left\{ \begin{bmatrix} w \\ u \end{bmatrix} : w \in W,\ u \in U \right\},$$

which becomes a normed linear space with the norm

$$\left\| \begin{bmatrix} w \\ u \end{bmatrix} \right\| = \sqrt{\|w\|_W^2 + \|u\|_U^2},$$

where $\|\cdot\|_W$ and $\|\cdot\|_U$ are the norms of W and U.

II. Problem Formulation

A. Plant and controller

Consider an uncertain linear time-invariant system:

$$\Sigma_P : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases}$$

where $x(t) \in \mathbb{R}^{n_s}$, $u(t) \in \mathbb{R}^{n_u}$, and $y(t) \in \mathbb{R}^{n_y}$ are the state, input, and output of the plant, respectively. This plant Σ_P is connected with a digital controller Σ_C through a zero-order hold and a sampler with period h:

$$\Sigma_C : \begin{cases} \xi[k+1] = A_c\xi[k] + B_c y[k] \\ \quad u[k] = C_c \xi[k]. \end{cases} \tag{1}$$

We place the following assumptions on the zero-order hold and the sample:

Assumption 2.1 (Periodic update of zero-order hold): The control input $u(t)$ is periodically generated through the zero-order hold with period h:

$$u(t) = u[k] \qquad (kh \le t < (k+1)h).$$

Assumption 2.2 (Perturbed sampling times): The sensor samples the plant output $y(t)$ at a perturbed sampling time $s_k = kh + \Delta_k$, where $\Delta_k \in [\underline{\Delta}, \overline{\Delta}] \in (-h, h)$ is the clock offset at time $t = kh$. The output y_k received by the controller is therefore defined by

$$y[k] := y(s_k) = Cx(kh + \Delta_k).$$

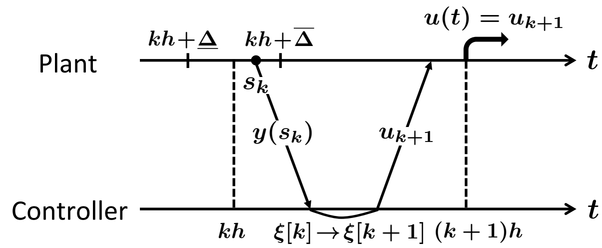

Fig. 1: Timing diagram.

Figure 1 illustrates the timing diagram.

B. Delayed impulsive system

We extend the discrete-time state $\xi[k]$ to a piecewise constant function $\xi(t)$ in the following way:

$$\xi(t) := \xi[k] \qquad (kh \le t < (k+1)h).$$

Then the closed-loop system Σ_{cl} can be represented as a delayed impulsive system

$$\Sigma_{cl} : \begin{cases} \dot{z}(t) = \begin{bmatrix} A & BC_c \\ 0 & 0 \end{bmatrix} z(t), \quad (kh \le t < (k+1)h) \\ z(kh) = \begin{bmatrix} I & 0 \\ 0 & A_c \end{bmatrix} z(kh^-) + \begin{bmatrix} 0 & 0 \\ B_cC & 0 \end{bmatrix} z(s_{k-1}), \end{cases} \tag{2}$$

where the closed-loop state z is defined by

$$z(t) := \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}.$$

For the delayed impulsive system Σ_{cl}, the initial state z_0 can be defined by

$$z_0 := \begin{bmatrix} x_0 \\ \xi_0 \end{bmatrix} \in \mathcal{W}_{2h+\underline{\Delta}} \oplus \mathcal{U}_h,$$

where $x_0(\theta) := x(\theta)$ for all $\theta \in [-(2h+\underline{\Delta}), 0]$ and $\xi_0(\theta) := \xi(\theta) = \xi[-1]$ for all $\theta \in [-h, 0]$. In fact, for all $t \in [kh, (k+1)h)$, the plant dynamics is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BC_c\xi[k] \\ &= Ax(t) + BC_c(A_c\xi(t-h) + B_c y[k]) \\ &= Ax(t) + BC_cB_cCx(t-(t-s_{k-1})) + BC_cA_c\xi(t-h), \end{aligned} \tag{3}$$

and the delay $t - s_{k-1}$ satisfies $t - s_{k-1} \le 2h + \underline{\Delta}$ for all $t \in [kh, (k+1)h)$. In what follows, we omit the subscripts $2h + \underline{\Delta}$ of $\mathcal{W}_{2h+\underline{\Delta}}$ and h of $\mathcal{U}_h$ for simplicity of notation.

Before stating our control problem, we define exponential stability for the closed-loop system Σ_{cl} in (2).

Definition 2.3: The delayed impulsive system Σ_{cl} in (2) is exponentially stable with decay rate $\gamma > 0$ if there exists $\Omega \ge 1$ such that $\|z(t)\| \le \Omega e^{-\gamma t}\|z_0\|$ for every $t \ge 0$ and for every $z_0 \in \mathcal{W} \oplus \mathcal{U}$.

In this paper, we study the following problem:

Problem 2.4: Let $\mathcal{P}$ be the set to which the matrices (A, B, C) belong. Given a controller (A_c, B_c, C_c) and a clock-offset bound $[\underline{\Delta}, \overline{\Delta}]$, determine whether the closed-loop system Σ_{cl} in (2) is exponentially stable for all system matrices $(A, B, C) \in \mathcal{P}$, and time-varying clock offsets $\Delta_k \in [\underline{\Delta}, \overline{\Delta}]$.

III. Main Results

First, we consider the case when the system matrices A and C have uncertainty.

Assumption 3.1 (Polytopic uncertainty for A and C): The system matrices A and C satisfy

$$\begin{bmatrix} A \\ C \end{bmatrix} \in \mathcal{P} := \left\{ \sum_{i=1}^{n} \alpha_i \begin{bmatrix} A_i \\ C_i \end{bmatrix} :\ \alpha_i \geq 0,\ \ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Theorem 3.2: Let Assumptions 2.1, 2.2, and 3.1 hold. Fix $\gamma > 0$. The closed-loop system is exponentially stable with decay rate $\gamma/2$ if there exist positive definite matrices P_i, $R_{k,i}$ ($k = 1, \ldots, 4, i = 1, \ldots, n$), Z_i, X_i, and Q_i and (not necessarily symmetric) matrices $N_{k,i}$ ($k = 1, \ldots, 4, i = 1, \ldots, n$), V_l, U_l, and W_l ($l = 1, 2$) such that

$$\begin{bmatrix} \Gamma_{1,i} & \Lambda_{1,i} \\ \star & \Xi_{1,i} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \Gamma_{2,i} & \Lambda_{2,i} \\ \star & \Xi_{2,i} \end{bmatrix} \geq 0, \tag{4}$$

where the matrices Γ, Λ, and Ξ are defined by (A) with $E_1 := [I\ 0\ 0\ 0\ 0] \in \mathbb{R}^{n_s \times (4n_s + n_c)}, \ldots, E_4 := [0\ 0\ 0\ I\ 0] \in \mathbb{R}^{n_s \times (4n_s+n_c)}$, $E_5 := [0\ 0\ 0\ 0\ I] \in \mathbb{R}^{n_c \times (4n_s+n_c)}$, $\tau_1 := h + \underline{\Delta}$, $\tau_2 := h$, $\tau_3 := h - \overline{\Delta}$, $\tau_4 := h + (\underline{\Delta} + \overline{\Delta})$, $\bar{\tau}_1 := 2h + \underline{\Delta}$, $\bar{\tau}_4 := \underline{\Delta} + \overline{\Delta}$,

$$\begin{aligned}
F_i &:= \begin{bmatrix} A_i & 0 & 0 & BC_cB_cC_i & BC_cA_c \end{bmatrix} \\
G_i &:= \begin{bmatrix} 0 & 0 & 0 & B_cC_i & A_c \end{bmatrix}, \quad \Pi_i := \begin{bmatrix} F_i \\ G_i \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
M_{0,i} :=\ & \mathrm{He}(N_{1,i}^\top (E_2 - E_4)) + \mathrm{He}(N_{2,i}^\top (E_1 - E_2)) \\
& + \mathrm{He}(N_{3,i}^\top (E_1 - E_3)) + \mathrm{He}(N_{4,i}^\top (E_3 - E_4)) \\
& - E_1^\top Z_i E_1 + E_3^\top (e^{-\gamma \tau_3} Z_i) E_3 \\
& + (E_1 - E_2)^\top X_i (E_1 - E_2) + E_5^\top (e^{-\gamma \tau_2} Q_i) E_5
\end{aligned}$$

$$M_{1,i} := M_{0,i} + \mathrm{He}\left(\begin{bmatrix} E_1 \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} \right) - \gamma \begin{bmatrix} E_1 \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
M_{2,i} :=\ & M_{0,i} + \mathrm{He}\left(\begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} \right) \\
& - \gamma \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} - \gamma h (E_1 - E_2)^\top X_i (E_1 - E_2)
\end{aligned}$$

$$\begin{aligned}
\Omega_{11,i} :=\ & \begin{bmatrix} I \\ 0 \end{bmatrix} (\bar{\tau}_1 R_{1,i} + \tau_2 R_{2,i} + \tau_3 R_{3,i} + \tau_4 R_{4,i}) \begin{bmatrix} I & 0 \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ I \end{bmatrix} Q_i \begin{bmatrix} 0 & I \end{bmatrix}
\end{aligned}$$

$$\Omega_{12,i} := \mathrm{He}\left(P_i \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} \right) + \gamma \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix}$$

$$\Omega_{1,i} := \Omega_{11,i} + \Omega_{12,i}, \quad \Omega_{2,i} := \Omega_{11,i}$$

$$\Upsilon_{1,i} := P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} 0 \\ E_5 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & hI \end{bmatrix} P_i \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$$

$$\Upsilon_{2,i} := P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} + h \begin{bmatrix} I \\ 0 \end{bmatrix} X_i (E_1 - E_2).$$

We will give the proof in Section IV by constructing a Lyapunov functional.

We next study the case when the system matrices A and B have uncertainty.

Assumption 3.3 (Polytopic uncertainty of A and B): The system matrices A and $B + B$ satisfy

$$\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P} := \left\{ \sum_{i=1}^{n} \alpha_i \begin{bmatrix} A_i & B_i \end{bmatrix} :\ \alpha_i \geq 0,\ \ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Similarly to Theorem 3.2 we can obtain the following result under Assumption 3.3.

Theorem 3.4: Let Assumptions 2.1, 2.2, and 3.3 hold. The closed-loop system is exponentially stable with decay rate $\gamma/2$ if there exist positive definite matrices P_i, $R_{k,i}$ ($k = 1, \ldots, 4, i = 1, \ldots, n$), Z_i, X_i, and Q_i and (not necessarily symmetric) matrices $N_{k,i}$ ($k = 1, \ldots, 4, i = 1, \ldots, n$), V_l, U_l, and W_l ($l = 1, 2$) such that LMIs (4) are feasible with the matrices Γ, Λ, and Ξ are defined by (A), where E, τ, $\bar{\tau}$, and $M_{0,i}$ are defined as in Theorem 3.2 and

$$\begin{aligned}
F_i &:= \begin{bmatrix} A_i & 0 & 0 & B_iC_cB_cC & B_iC_cA_c \end{bmatrix} \\
G &:= \begin{bmatrix} 0 & 0 & 0 & B_cC & A_c \end{bmatrix}, \quad \Pi_i := F_i
\end{aligned}$$

$$\begin{aligned}
M_{1,i} :=\ & M_{0,i} - G^\top Q_i G - \mathrm{He}\left(\begin{bmatrix} 0 \\ G - E_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix} \right) \\
& - \gamma \begin{bmatrix} E_1 \\ hG \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
M_{2,i} :=\ & M_{0,i} - G^\top Q_i G - \mathrm{He}\left(\begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} 0 \\ G - E_5 \end{bmatrix} \right) \\
& - \gamma \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix}
\end{aligned}$$

$$\Omega_{1,i} := \bar{\tau}_1 R_{1,i} + \tau_2 R_{2,i} + \tau_3 R_{3,i} + \tau_4 R_{4,i},\ \ \Omega_{2,i} := \Omega_{1,i}$$

$$\Upsilon_{1,i} := \begin{bmatrix} I \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hG \end{bmatrix}$$

$$\Upsilon_{2,i} := \begin{bmatrix} I \\ 0 \end{bmatrix}^\top P_i \begin{bmatrix} E_1 \\ hE_5 \end{bmatrix} + hX_i(E_1 - E_2).$$

This theorem can be proved in the same way as Theorem 3.2; see also Section IV.

Remark 3.5: If the controller is composed of a Luenberger-type observer and a feedback gain, then the controller parameters A_c, B_c, C_c in (1) are given by

$$A_c = A_d + B_d K + LC_d,\ \ B_c = -L,\ \ C_c = K,$$

where K and L are a feedback gain and an observer gain, respectively.

Remark 3.6: If we consider the state feedback case, then the state estimator is given by

$$\xi[k+1] = A_d x(s_k) + B_d u[k],$$

where $A_d := e^{Ah}$ and $B_d := \int_0^h e^{A(h-\tau)} d\tau B$. and we set the control input $u[k] = K\xi[k]$. Note that if the actual sampling

$$\Gamma_{l,i} := \begin{bmatrix} M_{l,i} - \mathrm{He}(\Pi_i^\top V_l) & -\Pi_i^\top U_1 & V_l^\top - \Upsilon_{l,i}^\top - \Pi_i^\top W_l \\ \star & \mathrm{He}(U_l) - \Omega_{l,i} & 0 \\ \star & \star & \mathrm{He}(W_l) \end{bmatrix} \qquad (l = 1, 2)$$

$$\Lambda_{1,i} := \begin{bmatrix} \tau_1 N_{1,i}^\top & \tau_2 N_{2,i}^\top & \tau_3 N_{3,i}^\top & \tau_4 N_{4,i}^\top \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Lambda_{2,i} := \begin{bmatrix} \tau_1 N_{1,i}^\top & \tau_3 N_{3,i}^\top & \bar{\tau}_4 N_{4,i}^\top \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{(A)}$$

$$\Xi_{1,i} := \mathrm{diag}(\tau_1 e^{-\gamma \bar{\tau}_1} R_{1,i}, \tau_2 e^{-\gamma \tau_2}(R_{1,i} + R_{2,i}), \tau_3 e^{-\gamma \tau_3} R_{3,i}, \tau_4 e^{-\gamma \bar{\tau}_1} R_{4,i})$$

$$\Xi_{2,i} := \mathrm{diag}(\tau_1 e^{-\gamma \bar{\tau}_1} R_{1,i}, \tau_2 e^{-\gamma \tau_3} R_{3,i}, \bar{\tau}_4 e^{-\gamma \bar{\tau}_1} R_{4,i})$$

time s_k satisfies $s_k = kh$, then $\xi[k+1] = x((k+1)h)$. Under Assumption 3.3, if we use F_i and G_i defined by

$$F_i := \begin{bmatrix} A_i & 0 & 0 & B_i K A_d & B_i K B_d K \end{bmatrix}$$
$$G_i := \begin{bmatrix} 0 & 0 & 0 & A_d & B_d K \end{bmatrix}$$

then the counterpart of Theorem 3.4 can be obtained in the state feedback case.

Remark 3.7: Since the state equation of the plant is given by (3) for all $t \in [kh, (k+1)h)$, we have a quadratic term BC_cB_cC with respect to the matrices B and C. In this paper, we therefore do not consider the case when B and C have uncertainty.

IV. Proof of Main Theorems

A. Preliminaries

Define $\rho_1(t) := t - s_{k-1}$ and $\rho_2(t) := t - kh$ for all $t \in [kh, (k+1)h)$, and define their supremums

$$\rho_{1,\max} := \sup_{t \ge 0} \rho_1(t), \quad \rho_{2,\max} := \sup_{t \ge 0} \rho_2(t) = h.$$

Define a function v by

$$v(t) := \int_{t-h}^{t} \xi(s) ds = (h - \rho_2(t))\xi[k-1] + \rho_2(t)\xi[k]$$

for every $t \in [kh, (k+1)h)$.

For each $t \ge 0$, define $x_t(\theta) := x(t+\theta)$ for all $\theta \in [-(2h + \underline{\Delta}), 0]$ and define $\xi_t(\theta) := \xi(t+\theta)$ for all $\theta \in [-h, 0]$. Since $\dot{x}_t$ is integrable in $[-h, 0]$ from the plant linearity, we have $x_t \in \mathcal{W}$. Also, since ξ_t is piecewise constant, it follows that $\xi_t \in \mathcal{U}$.

B. Lyapunov functional

Define a Lyapunov functional V by

$$\begin{aligned} &V(\rho_1(t), \rho_2(t), x_t, \xi_t) \\ &\quad := V_{c,d}(x_t, \xi_t) + V_c(\rho_1(t), \rho_2(t), x_t) + V_d(\xi_t) \end{aligned}$$

where $V_{c,d}$ is a Lyapunov functional for both of the continuous-time state x and the discrete-time state ξ, V_c is for the continuous-time state x, and V_d is for the discrete-time state ξ. These Lyapunov functionals are defined by $V_{c,d} := V_1$, $V_c := \sum_{i=2}^{8} V_i$, and $V_d := V_9$ with

$$V_1 := \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^\top P \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

$$V_2 := \int_{t-\rho_1(t)}^{t} (\rho_{1,\max} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_1 \dot{x}(s) ds$$

$$V_3 := \int_{t-\rho_2(t)}^{t} (\rho_{2,\max} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_2 \dot{x}(s) ds$$

$$V_4 := \int_{t-(h-\overline{\Delta})}^{t} ((h - \overline{\Delta}) - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_3 \dot{x}(s) ds$$

$$V_5 := \int_{t-\rho_1(t)}^{t-(h-\overline{\Delta})} (\rho_{1,\max} - t + s) e^{\gamma(s-t)} \dot{x}(s)^\top R_4 \dot{x}(s) ds$$

$$V_6 := (\rho_{1,\max} - (h - \overline{\Delta})) \int_{t-(h-\overline{\Delta})}^{t} e^{\gamma(s-t)} \dot{x}(s)^\top R_4 \dot{x}(s) ds$$

$$V_7 := \int_{t-(h-\overline{\Delta})}^{t} e^{\gamma(s-t)} x(s)^\top Z x(s) ds$$

$$V_8 := (\rho_{1,\max} - \rho_1(t))(x(t) - x(kh))^\top X (x(t) - x(kh))$$

$$V_9 := \int_{t-h}^{t} e^{\gamma(s-t)} \xi(s)^\top Q \xi(s) ds.$$

Note that we employ a continuous-time Lyapunov functional V for the stability analysis of the impulsive system Σ_{cl} in (2) that has both the continuous-time state x and the discrete-time state ξ. The Lyapunov functional V_c is used for the continuous-time state x and is inspired by [24].

On the other hand, for the discrete-time state ξ, we employ the Lyapunov functional V_d. In fact, for the autonomous system $\xi[k] = A_d \xi[k-1]$ and $\gamma = 0$, $V_d = V_9$ satisfies

$$\dot{V}_d(\xi_t) = \xi[k-1]^\top (A_d^\top Q A_d - Q) \xi[k-1]^\top$$

for all $t \in [kh, (k+1)h)$. Therefore, $\dot{V}_d < 0$ if and only if the discrete-time Lyapunov inequality $A_d^\top Q A_d - Q < 0$ holds.

C. Proof

As in the standard Lyapunov-based approach, we prove Theorem 3.2 by showing the positive definiteness of V and the negative definiteness of $\dot{V}$. We shall give the proofs of the following two lemmas in the appendix.

We first shows that V is positive definite if P, R_i, Z, X, and Q are positive definite.

Lemma 4.1: If the matrices P, R_i $(i = 1, \ldots, 4)$, Z, X, Q are positive definite, then there exist positive constants c_1 and c_2 such that

$$c_1 \left\| \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\|^2 < V(\rho_1(t), \rho_2(t), x_t, \xi_t) < c_2(\|x_t\|^2_{\mathcal{W}} + \|\xi_t\|^2_{\mathcal{U}}).$$

for all $t \geq 0$.

Next we show the negative-definiteness of $\dot{V}$ under the assumptions in Theorem 3.2.

Lemma 4.2: Let the assumptions in Theorem 3.2 hold. Define positive definite matrices P by $P := \sum_{i=1}^n P_i$ and R_k, Z, Q in a similar way. Then the time derivative $\dot{V}$ of V along the trajectory of Σ_{cl} in (2) satisfies

$$\dot{V}(\rho_1(t), \rho_2(t), x_t, \xi_t) \leq -\gamma V(\rho_1(t), \rho_2(t), x_t, \xi_t)$$

for all $t \in [kh, (k+1)h)$ $(k = 0, 1, 2, \ldots)$.

We are now in a position to prove the main theorem, Theorem 3.2.

Proof of Theorem 3.2: Since V_3 and V_8 vanish at time $t = kh$ and other functionals are continuous at time $t = kh$ for each k, it follows that

$$V(\rho_1(kh), \rho_2(kh), x_{kh}, \xi_{kh}) \leq \lim_{t \nearrow kh} V(\rho_1(t), \rho_2(t), x_t, \xi_t).$$

From Lemmas 4.1 and 4.2, we therefore have

$$\begin{aligned} c_1 \left\| \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\|^2 &\leq V(\rho_1(t), \rho_2(t), x_t, \xi_t) \\ &\leq e^{-\gamma t} V(\rho_1(0), \rho_2(0), x_0, \xi_0) \\ &\leq c_2 e^{-\gamma t}(\|x_0\|^2_{\mathcal{W}} + \|\xi_0\|^2_{\mathcal{U}}). \end{aligned}$$

It follows that there exists a constant $c > 0$ satisfying

$$\left\| \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\| \leq c e^{-\gamma t/2} \|z_0\|$$

for all $t \geq 0$. Since, in particular, $v((k+1)h) = \xi_k = \xi(t)$ for all $t \in [kh, (k+1)h)$, we obtain

$$\|\xi(t)\| \leq c e^{-(k+1)\gamma h/2} \|z_0\| \leq c e^{-\gamma t/2} \|z_0\|$$

for all $t \in [kh, (k+1)h)$. Also

$$\|x(t)\| \leq c e^{-\gamma t/2} \|z_0\|$$

for all $t \geq 0$. Thus the closed-loop system is exponential stable with decay rate $\gamma/2$. ■

We can prove Theorem 3.4 with small modification; see also Appendix C.

V. Discretization of the Closed-loop System

In this section, we show the discretization of the closed-loop system and discuss its advantage and disadvantage.

In this paper, we represent the closed-loop system as an impulsive system Σ_{cl} in (2). On the other hand, the authors of [10], [12], have studied stability by discretizing the closed-loop system. If we discretize the closed-loop system as in [10], [12], then we have

$$\eta[k+1] = \begin{cases} A_1(\Delta_k)\eta[k] & \text{if } \Delta_k > 0 \\ A_2(\Delta_k)\eta[k] & \text{if } \Delta_k \leq 0, \end{cases} \tag{5}$$

where

$$\eta[k] := \begin{bmatrix} x(kh) \\ \xi[k] \\ \xi[k-1] \end{bmatrix}$$

$$A_1(\Delta) := \begin{bmatrix} \tilde{A}_d & \tilde{B}_d K & 0 \\ -L\tilde{C}_d e^{(A+\delta A)\Delta} & A_c - LH_+(\Delta)K & 0 \\ 0 & I & 0 \end{bmatrix}$$

$$A_2(\Delta) := \begin{bmatrix} \tilde{A}_d & \tilde{B}_d K & 0 \\ -L\tilde{C}_d e^{(A+\delta A)\Delta} & A_c & LH_-(\Delta)K \\ 0 & I & 0 \end{bmatrix}$$

and

$$\begin{aligned} \tilde{A}_d &= e^{(A+\delta A)h} \\ \tilde{B}_d &:= \int_0^h e^{(A+\delta A)(h-\tau)} d\tau (B + \delta B), \\ \tilde{C}_d &:= C + \delta C \\ A_c &:= A_d + LC_d + B_d K \\ H_+(\Delta) &:= (C + \delta C) \int_0^\Delta e^{(A+\delta A)\tau} d\tau (B + \delta B) \\ H_-(\Delta) &:= (C + \delta C) \int_0^\Delta e^{-(A+\delta A)\tau} d\tau (B + \delta B). \end{aligned}$$

Although the discretization of the closed-loop system allows us to employ a gridding and norm-bounded approach [29]–[31] and a stochastic approach [32]. However, these approaches involve the exponential of the matrix A, and therefore stability analysis becomes conservative if the matrix A is uncertain. On the other hand, the impulsive system representation (2) leads to LMI conditions in which the matrix A appears in an affine form.

Another approach for robust stability is the input-delay approach [20]–[22], [33] and the loop-functional approach [34], [35], but here we do not proceed along these lines.

VI. Numerical Example

Consider the following continuous-time system including uncertainty in the matrices A and C:

$$\begin{aligned} \dot{x}(t) &= (A + \delta A)x(t) + Bu(t) \\ y(t) &= (C + \delta C)x(t), \end{aligned}$$

where the nominal system matrices A, B, C are given by

$$A := \begin{bmatrix} -2.6 & 2.9 \\ 3.9 & 4.2 \end{bmatrix}, \quad B := \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} -3.5 & 4 \end{bmatrix}$$

and the uncertainties $\delta A, \delta C$ are

$$\begin{bmatrix} \delta A \\ \delta C \end{bmatrix} \in \{\alpha(-F_\lambda) + (1-\alpha)F_\lambda,\ 0 \leq \alpha \leq 1\}$$

$$F_\lambda := \begin{bmatrix} 0.05\lambda & \lambda \\ 0 & 0 \\ 0.05 & 0.025 \end{bmatrix}.$$

We took the nominal sampling period h to be $h = 0.05$ and the feedback gain K and the observer gain L to be

$$K = -\begin{bmatrix} 3.6549 & 7.6954 \end{bmatrix}, \qquad L = -\begin{bmatrix} 0.0807 \\ 0.2213 \end{bmatrix},$$

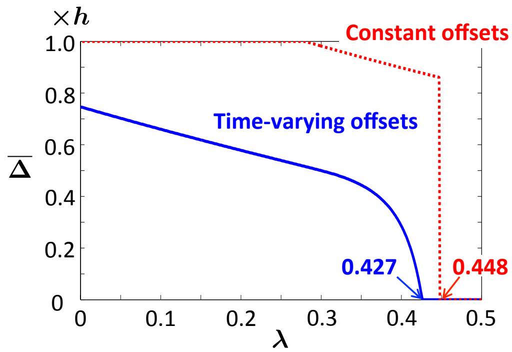

Fig. 2: Uncertainty parameter λ versus offset bound $[-\overline{\Delta}, \overline{\Delta}]$.

which are a linear quadratic regulator gain and a Kalman filter gain for the nominal plant (A, B, C), respectively. For simplicity, we consider a symmetric offset bound $[-\overline{\Delta}, \overline{\Delta}]$.

Fig. 2 illustrates the uncertainty parameter λ versus the clock-offset bound $[-\overline{\Delta}, \overline{\Delta}]$. The blue line is a lower bound on the allowable *time-varying* clock offsets, which was obtained by Theorem 3.2 with sufficiently small $\gamma > 0$, whereas the red dotted line indicates the exact bound on *constant* clock offsets that would be allowed by K and L without compromising the closed-loop stability. Note that the exact bound on constant offsets can be regarded as an upper bound on time-varying offsets. We can obtain the exact bound on constant offsets from iterative calculations of the eigenvalues of the discretized closed-loop system in (5).

The bound on allowable time-varying offsets decreases linearly when $0 \leq \lambda < 0.3$, but it drops rapidly for $\lambda > 0.35$. Similarly, the exact bound on constant offsets decreases linearly from $\lambda = 0.29$, and the closed-loop system suddenly becomes unstable at $\lambda = 0.448$. Theorem 3.2 showed that in the case without clock offsets, the closed-loop system may be unstable for $\lambda > 0.427$. We see that the difference between 0.427 and 0.448 is the conservativeness of Theorem 3.2 for the stability analysis of systems with polytopic uncertainties (and no clock offsets).

VII. Conclusion

We have studied robust stability analysis for systems that has polytopic uncertainty and variable clock offsets. We have represented the closed-loop system as a delayed impulsive system. This representation has led to a sufficient LMI condition for robust stability by using a Lyapunov functional whose variables are the states of the continuous-time plant and the discrete-time controller. Future work involves constructing less conservative Lyapunov functionals and addressing the case when transmission delays are larger than a sampling period.

Appendix

A. Proof of Lemma 4.1

First, we have the following lower bound:

$$V(\rho_1(t), \rho_2(t), x_t, \xi_t) \geq V_1(x_t, \xi_t) \geq \lambda_{\min}(P) \left\| \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\|^2 .$$

Regarding an upper bound, we obtain

$$V_1(x_t, \xi_t) \leq \lambda_{\max}(P) \left(\|x_t\|_{\mathcal{W}}^2 + h^2 \|\xi_t\|_{\mathcal{U}}^2 \right),$$

and

$$V_2(\rho_1(t), x_t) \leq (2h + \underline{\Delta}) \lambda_{\max}(R_1) \|x_t\|_{\mathcal{W}}^2 .$$

We have similar bounds for $V_2, \dots, V_6$. Also,

$$V_7(x_t) \leq (h - \overline{\Delta}) \lambda_{\max}(Z) \|x_t\|_{\mathcal{W}}^2$$
$$V_9(\xi_t) \leq h \lambda_{\max}(Q) \|\xi_t\|_{\mathcal{U}}^2,$$

and

$$V_8(\rho_1(t), x_t) \leq 4(2h + \underline{\Delta}) \lambda_{\max}(X) \|x_t\|_{\mathcal{W}}^2 .$$

Thus we derive

$$V(\rho_1(t), \rho_2(t), x_t, \xi_t) \leq c(\|x_t\|_{\mathcal{W}}^2 + \|\xi_t\|_{\mathcal{U}}^2)$$

for some constant $c > 0$. ■

B. Proof of Lemma 4.2

First we give a key technical result for the construction of a Lyapunov functional dependent on the uncertainty parameter α in Assumption 3.1. This result is inspired by [18], [19].

Proposition B.1: For arbitrary symmetric matrices M and Ω and (not necessarily symmetric) matrices Υ, Π, and Λ, the LMI

$$M - \Pi^\top \Omega \Pi - \mathrm{He}(\Lambda^\top \Upsilon) \geq 0 \tag{6}$$

is feasible, if there exist matrices V, U, and W such that

$$\begin{bmatrix} M - \mathrm{He}(\Lambda^\top V) & -\Pi^\top U & V^\top - \Upsilon^\top - \Lambda^\top W \\ \star & \mathrm{He}(U) - \Omega & 0 \\ \star & \star & \mathrm{He}(W) \end{bmatrix} \geq 0. \tag{7}$$

Furthermore, if $\Omega > 0$, then "only if" also holds.

Proof: ($\Leftarrow$) Let us denote by Θ the matrix in the left-hand side of (7). Since

$$\begin{bmatrix} I & \Pi^\top & \Lambda^\top \end{bmatrix} \Theta \begin{bmatrix} I \\ \Pi \\ \Lambda \end{bmatrix} = M - \Pi^\top \Omega \Pi - \mathrm{He}(\Lambda^\top \Upsilon),$$

it follows that (7) leads to (6).

($\Rightarrow$) Additionally, assume that $\Omega > 0$. Suppose that the LMI (6) is feasible. Then we have from the Schur complement formula that

$$\begin{bmatrix} M - \mathrm{He}(\Lambda^\top \Upsilon) & -\Pi^\top \Omega \\ \star & \Omega \end{bmatrix} \geq 0.$$

Hence (7) holds with $V = \Upsilon$, $U = \Omega$, and $W = 0$. ■

As we see the definitions of $\Omega_{1,i}$ and $\Omega_{2,i}$ in Theorem 3.2, $\Omega_{2,i}$ is positive definite, but $\Omega_{1,i}$ may not. On the other hand, the counterparts $\bar{\Omega}_{1,i}, \bar{\Omega}_{2,i}$ in Corollary 3.4 are both positive definite.

Proof of Lemma 4.2: Define ζ by

$$\zeta(t) := \begin{bmatrix} x(t) \\ x(kh) \\ x(t - (h - \overline{\Delta})) \\ x(s_{k-1}) \\ \xi[k-1] \end{bmatrix}$$

for $t \in [kh,(k+1)h)$. Define $\tilde{A} := A+\delta A$ and $\tilde{C} := C+\delta C$. Also set

$$F := \begin{bmatrix}\tilde{A} & 0 & 0 & BC_cB_c\tilde{C} & BC_cA_c\end{bmatrix} \tag{8}$$

$$G := \begin{bmatrix}0 & 0 & 0 & B_c\tilde{C} & A_c\end{bmatrix} \tag{9}$$

From the system dynamics, it follows that

$$\xi_k = A_c\xi_{k-1} + B_c\tilde{C}x(s_{k-1}),$$

and $v = ((h-\rho_2)E_5+\rho_2 G)\zeta$. Hence, for every $t \in [kh,(k+1)h)$, we have $\dot{v}(t) = (G-E_5)\zeta(t)$, and from (3), x satisfies $\dot{x}(t) = F\zeta(t)$ for all $t \in [kh,(k+1)h)$. Hence

$$\dot{V}_1 + \gamma V_1 = 2\zeta^\top \begin{bmatrix} F \\ G-E_5\end{bmatrix}^\top P \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}\zeta + \gamma\zeta^\top \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}^\top P \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}\zeta.$$

Also, we obtain

$$\dot{V}_9 + \gamma V_9 = \zeta^\top G^\top QG\zeta - \zeta^\top E_5^\top (e^{-\gamma h}Q)E_5\zeta.$$

Since the time-derivative of V_c for the continuous-time state x, namely, $V_2,\ldots,V_8$ can be obtained in the same way as in [24], we omit the details for brevity.

Using the fact that $\rho_1 = \rho_2 + (kh - s_{k-1})$ and $\rho_{1,\max} - \rho_1 = h - \rho_2$, we can remove ρ_1 and $\rho_{1,\max}$ from $\dot{V}$ and obtain

$$\dot{V} + \gamma V \le \zeta^\top(\Psi_1(\rho_2) + \gamma\Psi_2(\rho_2))\zeta,$$

where

$$\Psi_1(\rho_2) := \Phi_1 + \rho_2\Phi_2 + (\rho_2 + \underline{\Delta} + \overline{\Delta})\Phi_3 + (h-\rho_2)\Phi_4 + \mathrm{He}\left(\begin{bmatrix} F \\ G-E_5\end{bmatrix}^\top P \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}\right)$$

$$\Psi_2(\rho_2) := \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}^\top P \begin{bmatrix} E_1 \\ (h-\rho_2)E_5+\rho_2 G\end{bmatrix}$$

with $\tau_1 := h + \underline{\Delta},\ \tau_2 := h,\ \tau_3 := h - \overline{\Delta},\ \tau_4 := h + (\underline{\Delta} + \overline{\Delta}),\ \bar{\tau}_1 := 2h + \underline{\Delta},\ \bar{\tau}_4 := \underline{\Delta} + \overline{\Delta}$,

$$\begin{aligned}
\Phi_1 &:= \Phi_{1,1} + \Phi_{1,2}\\
\Phi_2 &:= N_2^\top (e^{-\gamma\tau_2}(R_1+R_2))^{-1}N_2\\
\Phi_3 &:= N_4^\top (e^{-\gamma\bar{\tau}_1}R_4)^{-1}N_4\\
\Phi_4 &:= \mathrm{He}(F^\top X(E_1-E_2)) + \gamma(E_1-E_2)^\top X(E_1-E_2)\\
\Phi_{1,1} &:= \tau_1 N_1^\top (e^{-\gamma\bar{\tau}_1}R_1)^{-1}N_1\\
&\quad - \mathrm{He}(N_1^\top(E_2-E_4)) - \mathrm{He}(N_2^\top(E_1-E_2))\\
&\quad + \tau_3 N_3^\top (e^{-\gamma\tau_3}R_3)^{-1}N_3\\
&\quad - \mathrm{He}(N_3^\top(E_1-E_3)) - \mathrm{He}(N_4^\top(E_3-E_4))\\
&\quad + E_1^\top ZE_1 - E_3^\top(e^{-\gamma\tau_3}Z)E_3\\
&\quad - (E_1-E_2)^\top X(E_1-E_2) - E_5^\top(e^{-\gamma\tau_2}Q)E_5\\
\Phi_{1,2} &:= F^\top(\bar{\tau}_1 R_1 + \tau_2 R_2 + \tau_3 R_3 + \tau_4 R_4)F + G^\top QG
\end{aligned}$$

In order to obtain $\dot{V} + \gamma V \le 0$ for all $t \in [kh,(k+1)h)$, it suffices that the matrix inequality

$$\Psi_1(\rho_2) + \gamma\Psi_2(\rho_2) \le 0$$

is feasible for all $t \in [kh,(k+1)h)$. Although ρ_2 of Ψ_2 does not appear in an affine form, by using the Schur complement formula, we see that, in order to derive $\dot{V} + \gamma V \le 0$ for all $t \in [kh,(k+1)h)$, it suffices that

$$\Psi_1(h) + \gamma\Psi_2(h) \le 0 \tag{10}$$

$$\Psi_1(0) + \gamma\Psi_2(0) \le 0. \tag{11}$$

In what follows, we obtain a sufficient LMI condition for (10) to hold. The same approach can be applied to (11), so we omit the details of (11) for brevity.

Define

$$\begin{aligned}
M_1 &:= -(\Phi_{1,1} + \tau_2\Phi_2 + \tau_4\Phi_3) + \mathrm{He}\left(\begin{bmatrix}E_1\\0\end{bmatrix}^\top P\begin{bmatrix}0\\E_5\end{bmatrix}\right) - \gamma\begin{bmatrix}E_1\\0\end{bmatrix}^\top P\begin{bmatrix}E_1\\0\end{bmatrix}\\
\Omega_{11} &:= \begin{bmatrix}I\\0\end{bmatrix}\left(\bar{\tau}_1 R_1 + \tau_2 R_2 + \tau_3 R_3 + \tau_4 R_4\right)\begin{bmatrix}I & 0\end{bmatrix} + \begin{bmatrix}0\\I\end{bmatrix}Q\begin{bmatrix}0 & I\end{bmatrix}\\
\Omega_{12} &:= \mathrm{He}\left(P\begin{bmatrix}0 & 0\\0 & hI\end{bmatrix}\right) + \gamma\begin{bmatrix}0 & 0\\0 & hI\end{bmatrix}P\begin{bmatrix}0 & 0\\0 & hI\end{bmatrix}\\
\Omega_1 &:= \Omega_{11} + \Omega_{12}\\
\Upsilon_1 &:= P\begin{bmatrix}E_1\\0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & hI\end{bmatrix}P\begin{bmatrix}0\\E_5\end{bmatrix} + \gamma\begin{bmatrix}0 & 0\\0 & hI\end{bmatrix}P\begin{bmatrix}E_1\\0\end{bmatrix}\\
\Pi_1 &:= \begin{bmatrix}F\\G\end{bmatrix}, \quad \Lambda_1 := \begin{bmatrix}F\\G\end{bmatrix}.
\end{aligned}$$

We have that (10) is feasible if and only if the following matrix inequality is feasible:

$$M_1 - \Pi_1^\top\Omega_1\Pi_1 - \mathrm{He}(\Lambda_1^\top\Upsilon_1) \ge 0. \tag{12}$$

Since the inequality (12) has the quadratic term of Π_1, in order to address the polytopic uncertainty in Π_1, we use Proposition B.1, which shows that the above inequality is feasible if

$$\begin{bmatrix} M_1 - \mathrm{He}(\Lambda_1^\top V_1) & -\Pi_1^\top U_1 & V_1^\top - \Upsilon_1^\top - \Lambda_1^\top W_1\\ \star & \mathrm{He}(U_1) - \Omega_1 & 0\\ \star & \star & \mathrm{He}(W_1)\end{bmatrix} \tag{13}$$

is positive semi-definite for some matrices V_1, U_1, W_1. Note that M_1 has the nonlinear term N_{all} defined by

$$\begin{aligned}
N_{all} :=& - \tau_1 N_1^\top(e^{-\gamma\bar{\tau}_1}R_1)^{-1}N_1 - \tau_2 N_2^\top(e^{-\gamma\tau_2}(R_1+R_2))^{-1}N_2\\
&- \tau_3 N_3^\top(e^{-\gamma\tau_3}R_3)^{-1}N_3 - \tau_4 N_4^\top(e^{-\gamma\bar{\tau}_1}R_4)^{-1}N_4,
\end{aligned}$$

Therefore, by using the Schur Complement formula, we obtain the following equivalent LMI:

$$\begin{bmatrix}\Gamma_1 & \Theta_1\\ \star & \Xi_1\end{bmatrix} \ge 0, \tag{14}$$

where Γ_1 is defined by a matrix in (13) and Θ_1 and Ξ_1 are defined by

$$\begin{aligned}\Theta_1 &:= \begin{bmatrix} \tau_1 N_1^\top & \tau_2 N_2^\top & \tau_3 N_3^\top & \tau_4 N_4^\top \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \Xi_1 &:= \operatorname{diag}(\tau_1 e^{-\gamma\bar{\tau}_1} R_1, \tau_2 e^{-\gamma\tau_2}(R_1 + R_2), \\ &\qquad \tau_3 e^{-\gamma\tau_3} R_3, \tau_4 e^{-\gamma\bar{\tau}_1} R_4).\end{aligned}$$

Define

$$\begin{aligned}\Pi_{1,i} &:= \begin{bmatrix} F_i \\ G_i \end{bmatrix}, \quad \Lambda_{1,i} := \begin{bmatrix} F_i \\ G_i \end{bmatrix} \\ F_i &:= \begin{bmatrix} A_i & 0 & 0 & BC_cB_cC_i & BC_cA_c \end{bmatrix} \\ G_i &:= \begin{bmatrix} 0 & 0 & 0 & B_cC_c & A_c \end{bmatrix}.\end{aligned}$$

Then we have $\Pi_1 = \sum_{i=1}^n \alpha_i \Pi_{1,i}$ and $\Lambda_1 = \sum_{i=1}^n \alpha_i \Lambda_{1,i}$ under Assumption 3.1. Note that the LMI (14) is affine with respect to Π_1, Λ_1, and matrix variables. Hence, if there exist positive definite matrices P_i, $R_{k,i}$, Z_i, X_i, and Q_i and (not necessarily symmetric) matrices $N_{k,i}$ such that

$$\begin{bmatrix} \Gamma_{1,i} & \Theta_{1,i} \\ \star & \Xi_{1,i} \end{bmatrix} \geq 0$$

for every $i = 1, \ldots, n$, where $\Gamma_{1,i}$, $\Theta_{1,i}$, $\Xi_{1,i}$ are defined by (A), then (14) is feasible with $P := \sum_{i=1}^n \alpha_i P_i$ and R_k, Z, X, Q, N_k ($k = 1, \ldots, 4$) defined in the same way. This completes the proof. ■

C. *Proof of Corollary 3.4:*

Since G in (9) has no uncertainty, we can transform (10) to (12) by using

$$\Pi_1 = \Lambda_1 = F \quad \text{instead of} \quad \Pi_1 = \Lambda_1 = \begin{bmatrix} F \\ G \end{bmatrix}.$$

A routine calculation leads to the desired result. The details are omitted for brevity. ■

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