Switching Between Stabilizing Controllers [3]

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Abstract

This paper deals with the problem of switching between several linear time-invariant (LTI) controllers—all of them capable of stabilizing a specific LTI process—in such a way that the stability of the closed-loop system is guaranteed for any switching sequence. We show that it is possible to find realizations for any given family of controller transfer matrices so that the closed-loop system remains stable, no matter how we switch among the controller. The motivation for this problem is the control of complex systems where conflicting requirements make a single LTI controller unsuitable.

Key words: Switched Systems; Impulse System; Hybrid Systems; Realization Theory; Stability Theory.

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1 Introduction

Given a finite set of matrices $\mathcal{A} := \{A_p : p \in \mathcal{P}\}$, consider the linear time-varying system

$$\dot{x} = A_\sigma x, \tag{1}$$

where σ denotes a piecewise constant "switching signal" taking values on $\mathcal{P}$. The following question has often been posed: "Under what conditions is the system (1) uniformly asymptotically stable for *every* piecewise constant switching signal σ?" [1–11]. In [5] it is shown that uniform asymptotic stability of (1) for every switching signal σ is equivalent to the existence of an induced norm $\|\cdot\|_*$ and a positive constant α such that

$$\|e^{At}\|_* \leq e^{-\alpha t}, \qquad \forall t \geq 0,\ \forall A \in \mathcal{A}.$$

In [9] it is shown that uniform asymptotic stability of (1) for every switching signal σ is also equivalent to the existence of a common Lyapunov function (not necessarily quadratic) for the family of linear time-invariant systems $\{\dot{z} = A_p z : p \in \mathcal{P}\}$. However, the proofs in [5,9] are not constructive and not amenable to test the stability of general switched system. In [2,7,10] are given simple algebraic conditions on the elements of $\mathcal{A}$, which are sufficient for the existence of a common quadratic Lyapunov function for the family of linear time-invariant systems $\{\dot{z} = A_p z : p \in \mathcal{P}\}$, and therefore for the uniform asymptotic stability of (1) for every switching signal σ. However, it is known that none of these conditions are necessary for the stability of the switched system. For more on this topic see [3,4,11] and references therein.

A simple and general test to check the uniform asymptotic stability of (1), for every switching signal σ, has eluded researchers for more than a decade. However, when systems like (1) arise in control problems, in general, the matrices in $\mathcal{A}$ have specific structures. In fact, these matrices are often obtained from the feedback connection of a fixed process with one of several controllers, and the switching signal σ determines which controller is in the feedback loop at each instant of time. One can then pose the question if, by appropriate choice of the realizations for the controllers, it is possible to make the system (1) uniformly asymptotically stable for every switching signal σ. This is precisely the question addressed in this paper. The motivation for this problem is the control of complex systems where conflicting requirements make a single linear time-invariant controller unsuitable. The reader is referred to [12] for a detailed discussion on the tradeoffs that arise when a single linear controller is used to meet multiple performance specifications (e.g., involving bandwidth, time-response, robustness with respect to modeling errors, etc.). Controller switching to improve the tradeoff in design objectives has been proposed in a few papers. In [13] a logic was devised to orchestrate the switching between several controllers, some with high-performance/low-robustness and others with low-performance/high-robustness. In [14], switching among PID controllers was used to achieve fast step-response without overshoot. In Section 5, we illustrate the use of switching in the control of the roll angle of an aircraft. We design two controllers: the first is slow but has good noise rejection properties, whereas the second is fast but very sensitive to measurement noise. By switching between the controllers, we are able to achieve good noise rejection when the noise is large and yet obtain a fast response when the noise is small (cf. Figure 6). The method used to implement the switching controller guarantees stability *regardless of the algorithm used to command the switching between the controllers*. This means that one can use simpleminded algorithms to switch between the two controllers, without fear of causing instability.

In this paper we assume that the process $\mathbf{P}$ to be controlled is modeled by a linear, time-invariant, stabilizable and detectable system of the form

$$\dot{x}_{\mathbf{P}} = Ax_{\mathbf{P}} + Bu, \qquad y = Cx_{\mathbf{P}}. \tag{2}$$

We take as given a family of controller transfer matrices $\mathcal{K} := \{\mathrm{K}_p : p \in \mathcal{P}\}$ with the property that, for each $p \in \mathcal{P}$, the feedback interconnection of (2) with any stabilizable and detectable realization of K_p is asymptotically stable. Let then $\dot{x}_p = A_p x_p$ denote the system that results from the pth such interconnection. The main result of this paper is to prove that if the controller realizations are chosen properly, then for any piecewise constant signal $\sigma : [0, \infty) \to \mathcal{P}$, the switched system

$$\dot{x} = A_\sigma x, \tag{3}$$

will be uniformly exponentially stable. That the stability of (3) should be controller realization dependent is not surprising, but the fact that there is actually a way to realize the controllers that is guaranteed to achieve stability for every σ perhaps is. The approach we use to establish this result relies on the fact that all the controller transfer matrices can be expressed using the Youla parameterization with a distinct value of the Youla parameter for each controller [15]. Switching between controllers can thus be reduced to switching between the corresponding values of the parameter. The same idea has been independently discovered by A. Packard [16] but was not published.

The Youla parameters used to represent the controller transfer matrices in $\mathcal{K}$ are stable transfer matrices. An important step in the overall controller realization procedure is to select realizations for the individual Youla parameters so that switching between them results in a stable time-varying system $\mathbf{S}(\sigma)$. There are two ways to accomplish this: The first is to develop realizations for the Youla parameters for which there is a common Lyapunov function. In the second, the state of $\mathbf{S}(\sigma)$ is reset at switching times resulting in a what is often called a "system with impulse effects." We show that both approaches are possible. The idea of reseting part of the controller state dates as far back as the 50s with the Clegg Integrator [17] and later with Horowitz and Rosenbaum's first-order reset elements (FORE) [18]. The reader is referred to [19] for more recent references on this form of "reset control," whose goal is to improve transient performance.

The problem addressed in this paper is precisely formulated in Section 2. In Section 3 we derive some basic results to study the stability of systems with impulse effects. These results are used in Section 4 to construct the desired realizations for the controller transfer functions: in 4.1 we motivate the construction by considering the simpler case of a single-input/single-output stable process and in 4.2 we address the general case. A simple illustrative example is presented in Section 5. Section 6 contains some concluding remarks and directions for future research. A preliminary version of the results in this paper was presented at the 12th Int. Symposium on the Mathematical Theory of Networks and Syst., St. Louis, MO, June 1996. These were subsequently improved in the PhD thesis [20].

2 Stable controller switching

The feedback configuration used in this paper is shown in Figure 1. In this

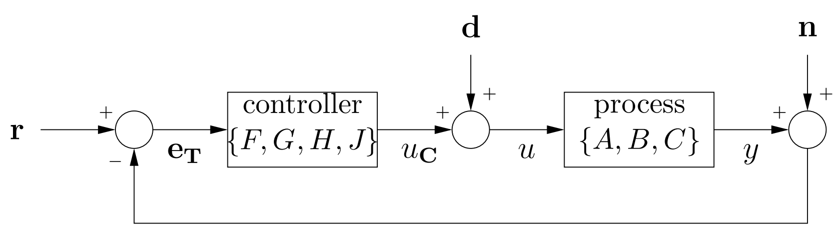


Fig. 1. Feedback configuration

figure u denotes the control input, y the process output, $\mathbf{r}$ a bounded reference signal, $\mathbf{d}$ an unknown but bounded input disturbance, and $\mathbf{n}$ unknown but bounded measurement noise. The process will be denoted by $\mathbf{P}$ and is assumed to be a multivariable linear time-invariant system with strictly proper transfer matrix $\mathrm{H}_{\mathbf{P}}$. We say that a given controller transfer matrix K *stabilizes* $\mathrm{H}_{\mathbf{P}}$ if, for any stabilizable and detectable realizations $\{A, B, C\}$ and $\{F, G, H, J\}$ of $\mathrm{H}_{\mathbf{P}}$ and K, respectively, the feedback connection shown in Figure 1 is asymptotically stable, i.e., all the poles of the matrix

$$\begin{bmatrix} A - BJC & BH \\ -GC & F \end{bmatrix} \tag{4}$$

have negative real part. We recall that a quadruple of matrices $\{A, B, C, D\}$ is called a *realization* for a transfer matrix H if $\mathrm{H}(s) = C(sI - A)^{-1}B + D$ for every $s \in \mathbb{C}$. When the matrix D is equal to zero one often writes simply that $\{A, B, C\}$ is a realization for H.

Consider now a finite set of controller transfer matrices $\mathcal{K} = \{\mathrm{K}_p : p \in \mathcal{P}\}$ each stabilizing the process transfer matrix $\mathrm{H}_{\mathbf{P}}$. The general problem under consideration is to build a "multi-controller" that effectively switches among the transfer functions in $\mathcal{K}$. In this context, a *multi-controller* is a dynamical system $\mathbf{C}(\sigma)$ with two inputs σ, $\mathbf{e_T}$ and one output $u_{\mathbf{C}}$. The input $\sigma : [0, \infty) \to \mathcal{P}$ is piecewise constant and is called a *switching signal*. While σ remains constant and equal to some $p \in \mathcal{P}$, $\mathbf{C}(\sigma)$ is required to behave as a linear time-invariant system with transfer function K_p from its input $\mathbf{e_T}$ to its output $u_{\mathbf{C}}$. The multi-controller design problem is nontrivial because we also require that all the closed-loop signals remain bounded for *every* possible switching signal in the set $\mathcal{S}$ of all piecewise constant switching signals. The times at which a signal $\sigma \in \mathcal{S}$ is discontinuous are called the *switching times of* σ. For simplicity of notation we take all signals in $\mathcal{S}$ to be continuous from above at switching times, i.e., if t_1 and t_2 are two consecutive switching times of $\sigma \in \mathcal{S}$ then σ is constant on $[t_1, t_2)$.

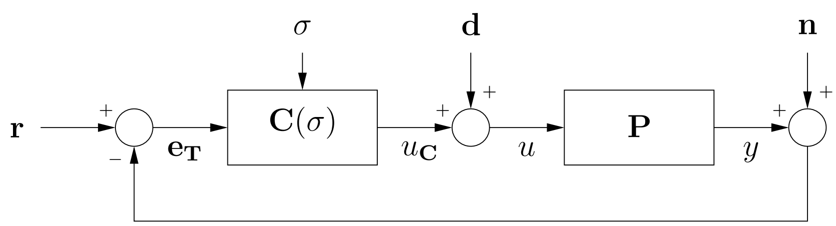


Fig. 2. Feedback connection between $\mathbf{P}$ and $\mathbf{C}(\sigma)$.

To build a multi-controller we start by selecting $n_{\mathbf{C}}$-dimensional stabilizable and detectable realizations $\{F_p, G_p, H_p, J_p\}$ for each K_p in $\mathcal{K}$. Over any open interval on which a switching signal $\sigma \in \mathcal{S}$ is constant, the multi-controller $\mathbf{C}(\sigma)$ is then defined by the following dynamical system

$$\dot{x}_{\mathbf{C}} = F_\sigma x_{\mathbf{C}} + G_\sigma \mathbf{e_T}\,, \qquad u_{\mathbf{C}} = H_\sigma x_{\mathbf{C}} + J_\sigma \mathbf{e_T}, \tag{5}$$

which possesses the desired transfer function from $\mathbf{e_T}$ to $u_{\mathbf{C}}$. By itself, (5) does not determine what happens to $x_{\mathbf{C}}$ at the switching times of σ. A rule must therefore be specified to determine the value of $x_{\mathbf{C}}$ immediately after a switching time. Such a rule takes the general form[4]

$$x_{\mathbf{C}}(t) = r\Big(x_{\mathbf{C}}(t^-); \sigma(t), \sigma(t^-)\Big),$$

[4] Here and in the sequel, given a signal z we denote by $z(t^-)$ the limit of $z(\tau)$ as $\tau \to t$ from below, i.e., $z(t^-) := \lim_{\tau \uparrow t} z(\tau)$. Without loss of generality we take $x_{\mathbf{C}}$ to be continuous from above at every point, i.e., $x_{\mathbf{C}}(t) = \lim_{\tau \downarrow t} x_{\mathbf{C}}(\tau)$.

where $r : \mathbb{R}^{n_\mathbf{C}} \times \mathcal{P} \times \mathcal{P} \to \mathbb{R}^{n_\mathbf{C}}$ is called a *reset map*. In this paper we restrict reset maps to be linear functions of $x_\mathbf{C}$, i.e.,

$$x_\mathbf{C}(t) = R_\mathbf{C}\Big(\sigma(t), \sigma(t^-)\Big)\, x_\mathbf{C}(t^-), \tag{6}$$

where the $R_\mathbf{C}(p,q) \in \mathbb{R}^{n_\mathbf{C} \times n_\mathbf{C}}$, $p, q \in \mathcal{P}$, are called *reset matrices*. Systems like the one defined by (5)–(6) are often called *systems with impulse effects* (c.f., [21,22] and references therein).

Consider now the feedback connection between $\mathbf{C}(\sigma)$ and $\mathbf{P}$ in Figure 2 and let $\{A, B, C\}$ denote a $n_\mathbf{P}$-dimensional stabilizable and detectable realization for the process transfer function $\mathrm{H}_\mathbf{P}$. Over any open interval on which $\sigma \in \mathcal{S}$ is constant, the feedback connection in Figure 2 corresponds to the dynamical system

$$\dot{x} = A_\sigma x + B_\sigma w \qquad\qquad y = Cx, \tag{7}$$

with $x := \begin{bmatrix} x'_\mathbf{P} & x'_\mathbf{C} \end{bmatrix}'$, $w := \begin{bmatrix} \mathbf{d}' & \mathbf{r}' - \mathbf{n}' \end{bmatrix}'$, $C := \begin{bmatrix} C & 0 \end{bmatrix}$, and

$$A_p := \begin{bmatrix} A - B\,J_p\,C & B\,H_p \\ -G_p\,C & F_p \end{bmatrix}, \qquad B_p := \begin{bmatrix} B & B\,J_p \\ 0 & G_p \end{bmatrix}, \qquad p \in \mathcal{P}; \tag{8}$$

whereas, at a switching time t,

$$x(t) = R\Big(\sigma(t), \sigma(t^-)\Big)\, x(t^-), \tag{9}$$

with

$$R(p,q) := \begin{bmatrix} I_{n_\mathbf{P}} & 0 \\ 0 & R_\mathbf{C}(p,q) \end{bmatrix}, \qquad\qquad p, q \in \mathcal{P}. \tag{10}$$

Since each transfer matrix in $\mathcal{K}$ stabilizes $\mathrm{H}_\mathbf{P}$, (7) is asymptotically stable for any constant $\sigma(t) = p \in \mathcal{P}$, $t \geq 0$. But, in general, this is not enough to guarantee that the state of (7)–(9) remains bounded for every $\sigma \in \mathcal{S}$. Examples of unstable behavior resulting from the switching amount stable systems are well known and can be found, e.g., in [23] or the recent survey [11].

The problem under consideration can then be summarized as follows: Given the family $\mathcal{K}$ of controller transfer functions, compute reset matrices and realizations for the transfer functions in $\mathcal{K}$ so that the state x of the closed-loop switched system (7)–(9) remains bounded for every switching signal $\sigma \in \mathcal{S}$ and every bounded piecewise continuous exogenous inputs $\mathbf{r}$, $\mathbf{n}$, and $\mathbf{d}$. We shall also require x to decay to zero, when $\mathbf{r} = \mathbf{d} = \mathbf{n} = 0$.

In this paper we assume that the set of controllers is finite just for simplicity. The finiteness assumption could be replaced by appropriate uniformity assumptions. For example, one could require compactness of $\mathcal{P}$ and continuity of the coefficients of the controller transfer matrices with respect to the parameter p.

3 Stability of systems with impulse effects

Consider the n-dimensional homogeneous linear system with impulse effects defined by

$$\dot{z} = A_\sigma z, \tag{11}$$

on intervals where the switching signal $\sigma \in \mathcal{S}$ remains constant, and by

$$z(t) = R\Big(\sigma(t), \sigma(t^-)\Big)\, z(t^-) \tag{12}$$

at each switching time t of σ. The solution to (11)–(12) can be written as

$$z(t) = \Phi(t, t_0; \sigma) z(t_0), \qquad\qquad t, t_0 \in \mathbb{R}, \tag{13}$$

where $\Phi(t, t_0; \sigma)$ denotes the *state-transition matrix of* (11)–(12) and is defined by

$$\Phi(t, t_0; \sigma) := e^{(t-t_m)A_{\sigma(t_m)}} \prod_{k=0}^{m-1} R\Big(\sigma(t_{k+1}), \sigma(t_k)\Big) e^{(t_{k+1}-t_k)A_{\sigma(t_k)}}.$$

Here $\{t_1, t_2, ..., t_m\}$ denote the switching times of σ in the interval $(t_0, t]$. The system (11)–(12) is called *exponentially stable, uniformly over* $\mathcal{S}$, if there exist positive constants c, λ such that [5], for every $\sigma \in \mathcal{S}$,

$$\|\Phi(t, t_0; \sigma)\| \leq c\, e^{-\lambda(t-t_0)}, \qquad\qquad \forall t, t_0 \geq 0. \tag{14}$$

State-transition matrices of systems with impulse effects share many of the properties of the usual state transition matrices for linear systems [6]. In particular, for any $\sigma \in \mathcal{S}$ and $\tau \in \mathbb{R}$, (i) $\Phi(\tau, \tau; \sigma) = I_n$, (ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, \tau; \sigma) = A_{\sigma(t)}\Phi(t, \tau; \sigma),$$

[5] Given a vector v and a matrix A we denote by $\|v\|$ and $\|A\|$ the Euclidean norm of v and the largest singular value of A, respectively.

[6] However, one should keep in mind that Φ does not share *all* properties of the usual state transition matrices, e.g., it may become singular.

for t in the interior of an interval on which $\sigma \in \mathcal{S}$ is constant, and (iii)

$$\Phi(t,\tau;\sigma) = R\Big(\sigma(t), \sigma(t^-)\Big)\Phi(t^-,\tau;\sigma),$$

for each switching time t. From the previous properties it is also straightforward to conclude that the variation of constants formula holds for non-homogeneous systems with impulse effects. In fact, the solution to the system defined by

$$\dot{x} = A_\sigma x + B_\sigma w \tag{15}$$

on intervals where $\sigma \in \mathcal{S}$ remains constant and by

$$x(t) = R\Big(\sigma(t), \sigma(t^-)\Big)\, x(t^-) \tag{16}$$

at each switching time t of σ, can be written as

$$x(t) = \Phi(t,t_0;\sigma)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau;\sigma)B_{\sigma(\tau)}w(\tau)d\tau, \qquad t, t_0 \in \mathbb{R}. \tag{17}$$

It is then straightforward to show that x will remain bounded for every $\sigma \in \mathcal{S}$ and bounded piecewise continuous w, as long as (11)–(12) is exponentially stable, uniformly over $\mathcal{S}$.

Suppose now that there exist symmetric, positive definite matrices $\{Q_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$, such that

$$Q_p A_p + A_p' Q_p < 0, \qquad\qquad p \in \mathcal{P}, \tag{18}$$

and

$$R(p,q)' Q_p R(p,q) \le Q_q, \qquad\qquad p,q \in \mathcal{P}. \tag{19}$$

Equation (18) guarantees that, on any interval where σ remains constant and equal to $p \in \mathcal{P}$, the positive definite Lyapunov-like function $V_p(z) := z'Q_p z$, decreases exponentially along solutions to (11). Indeed, on such an interval

$$\frac{\mathrm{d}}{\mathrm{d}t}V_p(z(t)) = z(t)'(Q_pA_p + A_p'Q_p)z(t) \le -2\lambda V_p(z(t)), \tag{20}$$

for sufficiently small $\lambda > 0$. Moreover, because of (19), when σ switches from $q := \sigma(t^-)$ to $p := \sigma(t)$, we have

$$\begin{aligned} V_p(z(t)) := z(t)'Q_p z(t) = z(t^-)'R(p,q)'Q_pR(p,q)z(t^-) \\ \le z(t^-)'Q_q z(t^-) =: V_q(z(t^-)). \end{aligned} \tag{21}$$

From (20)–(21) we then conclude that

$$V_{\sigma(t)}(z(t)) \le e^{-2\lambda(t-t_0)} V_{\sigma(t_0)}(z(t_0)), \qquad\qquad t \ge t_0, \tag{22}$$

along solutions to (11)–(12). Note that $V_{\sigma(t)}(z(t))$ may be discontinuous at switching times but its value will always decrease at these times because of (21). Since $V_{\sigma(t)}(z(t))$ is quadratic and the Q_p are positive definite, from (22) we actually conclude that

$$\|z(t)\| \leq ce^{-\lambda(t-t_0)}\|z(t_0)\|,$$

with $c := \max_{p,q\in\mathcal{P}} \sqrt{\|Q_p\|\|Q_q^{-1}\|}$. This and (13) prove that (14) holds true for every $\sigma \in \mathcal{S}$ and therefore (11)–(12) is exponentially stable, uniformly over $\mathcal{S}$. Similar analysis using multiple Lyapunov functions can be found, e.g., in [22,23], in the context of hybrid systems. The following can then be stated:

Lemma 1 *Assume that there exist symmetric matrices $\{Q_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$, for which* (18)–(19) *hold. Then the homogeneous system* (11)–(12) *is exponentially stable, uniformly over $\mathcal{S}$. Moreover, for every switching signal $\sigma \in \mathcal{S}$ and every bounded piecewise continuous signal w, the state x of the non-homogeneous system* (15)–(16) *is bounded.*

It is interesting to consider two special cases of the previous result. The first corresponds to a complete state reset, i.e., $R(p,q) = 0$, for all $p,q \in \mathcal{P}$. In this case, (19) is trivially true and the only requirement for the stability of the switched system is that each A_p be asymptotically stable. Note that this is enough for the existence of the positive definite matrices $\{Q_p : p \in \mathcal{P}\}$ for which (18) holds.

Another important case is the absence of state reset, i.e., when all the $R(p,q)$, $p,q \in \mathcal{P}$, are equal to the identity matrix. In this case, (19) actually requires all the Q_p to be the same because it demands both $Q_p \leq Q_q$ and $Q_q \leq Q_p$, for all $p,q \in \mathcal{P}$. The inequalities (18)–(19) then demand the existence of a common Lyapunov function for the family of linear time-invariant systems $\{\dot{z} = A_p z : p \in \mathcal{P}\}$. This is a well known sufficient condition for the exponential stability of the switched system (7). Later we will see that it is actually possible to always choose realizations for the controller so that a common Lyapunov function exists for the closed-loop systems. This avoids the need to reset the state of the controllers.

In the remaining of this paper we address the question of computing reset matrices and realizations for the transfer functions in $\mathcal{K}$ so that (18)–(19) hold for the closed-loop matrices A_p in (8). Because of Lemma 1, this will guarantee that, for every switching signal $\sigma \in \mathcal{S}$ and every bounded piecewise continuous exogenous signals $\mathbf{r}$, $\mathbf{n}$, and $\mathbf{d}$, the state x of (7)–(9) is bounded. Moreover, when $\mathbf{r} = \mathbf{d} = \mathbf{n} = 0$, x decays to zero exponentially fast with a rate of decay that is independent of σ. Before proceeding two remarks should be made about Lemma 1:

Remark 2 *The exponential stability of* (11)–(12) *guarantees that the sys-*

tem (7)–(9) *remains stable under small perturbations to the dynamics of the system. A detailed discussion of this issue for systems without impulsive effects can be found, e.g., in [24, Section 4.5]. It is straightforward to extend these results to the systems considered here.*

Remark 3 *In case the Lyapunov inequalities* (18) *were replaced by the Riccati inequalities*

$$Q_pA_p + A_p'Q_p + C_p'C_p + \gamma^{-2}Q_pB_pB_p'Q_p \leq 0, \qquad p \in \mathcal{P},$$

then we would actually by able to conclude that the $\mathcal{L}_2$ induced norm from w to

$$y := C_\sigma x$$

is no larger than γ, along trajectories of the switched system (11)–(12)*. This could be proved by showing that*

$$V_{\sigma(t)}(z(t)) := z(t)'Q_{\sigma(t)}z(t) + \int_0^t (\|y\|^2 - \gamma^2\|w\|^2)d\tau$$

is nonincreasing, and therefore that

$$\int_0^t (\|y\|^2 - \gamma^2\|w\|^2)d\tau \leq -z(t)'Q_{\sigma(t)}z(t) \leq 0,$$

for zero initial conditions. Analogous results could be derived to establish the dissipativeness of (11)–(12)*, as well as more general Integral Quadratic Constrains [25] that can be expressed in terms of linear and bilinear matrix inequalities [3].*

4 Realizations for controller transfer matrices

We now return to the problem formulated in Section 2. To motivate the approach we start by considering the case of a single-input/single-output asymptotically stable process.

4.1 Single-input/single-output stable process

Suppose we connect the process to a controller with transfer function K_p, $p \in \mathcal{P}$, as in Figure 1. The transfer function from $\mathbf{r}$ to $u_{\mathbf{C}}$ is then given by

$$\mathrm{S}_p := \frac{\mathrm{K}_p}{1 + \mathrm{K}_p\mathrm{H}_{\mathbf{P}}}. \tag{23}$$

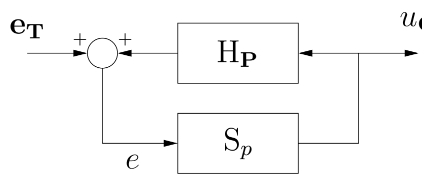


Fig. 3. Block diagram corresponding to equation (25). The transfer function from $\mathbf{e_T}$ to $u_{\mathbf{C}}$ is given by (24), which expresses K_p in terms of S_p (stable process case).

Since K_p stabilizes $\mathrm{H}_{\mathbf{P}}$, S_p must be asymptotically stable. From (23) we also conclude that

$$\mathrm{K}_p = \frac{\mathrm{S}_p}{1 - \mathrm{H}_{\mathbf{P}}\mathrm{S}_p}, \tag{24}$$

and therefore the transfer function K_p from $\mathbf{e_T}$ to $u_{\mathbf{C}}$ can be defined implicitly by the following system of equations[7] (cf. Figure 3).

$$u_{\mathbf{C}} = \mathrm{S}_p \circ e, \qquad\qquad e := \mathrm{H}_{\mathbf{P}} \circ u_{\mathbf{C}} + \mathbf{e_T}. \tag{25}$$

Since only S_p in (25) changes from controller to controller, this suggests a mechanism for switching among the controller transfer functions in $\mathcal{K}$:

(1) Pick stabilizable and detectable n_S-dimensional realizations $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ for each S_p, $p \in \mathcal{P}$, defined by (23).
(2) Define $\mathbf{S}(\sigma)$ to be the system with impulse effects defined by

$$\dot{x} = \bar{A}_\sigma x + \bar{B}_\sigma e, \qquad\qquad u_{\mathbf{C}} = \bar{C}_\sigma x + \bar{D}_\sigma e,$$

on intervals where σ is constant and by

$$x(t) = \bar{R}\Big(\sigma(t), \sigma(t^-)\Big)\, x(t^-),$$

at each switching time t of σ. For the time being we do not commit to a particular choice for the reset matrices $\{\bar{R}(p,q) : p, q \in \mathcal{P}\}$.
(3) Inspired by the implicit definition of K_p given by (25) (and the corresponding block diagram in Figure 3), we realize the switching controller as in Figure 4. This corresponds to the multi-controller in (5)–(6) with

$$F_p := \begin{bmatrix} A + B\bar{D}_p C & B\bar{C}_p \\ \bar{B}_p C & \bar{A}_p \end{bmatrix}, \qquad G_p := \begin{bmatrix} B\bar{D}_p \\ \bar{B}_p \end{bmatrix},$$
$$H_p := \begin{bmatrix} \bar{D}_p C & \bar{C}_p \end{bmatrix}, \qquad J_p := \bar{D}_p,$$

[7] Given a transfer matrix $\mathrm{H} : \mathbb{C} \to \mathbb{C}^{m \times n}$ and a piecewise constant signal $u : [0, \infty) \to \mathbb{R}^n$, $\mathrm{H} \circ u$ denotes the signal defined by the convolution of the impulse response of H with u.

and

$$R_{\mathbf{C}}(p,q) := \begin{bmatrix} I & 0 \\ 0 & \bar{R}(p,q) \end{bmatrix},$$

where $\{A, B, C\}$ is a stabilizable and detectable realization for $\mathrm{H}_{\mathbf{P}}$.

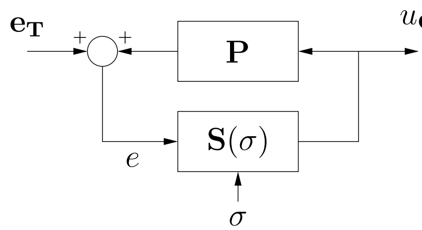


Fig. 4. Multicontroller $\mathbf{C}(\sigma)$ inspired by the implicit definition of K_p given by (25) and the corresponding block diagram in Figure 3 (stable process case).

Suppose now that we connect this multi-controller to the process as in Figure 5. Because the process is stable, no matter what $u_{\mathbf{C}}$ turns out to be, we have

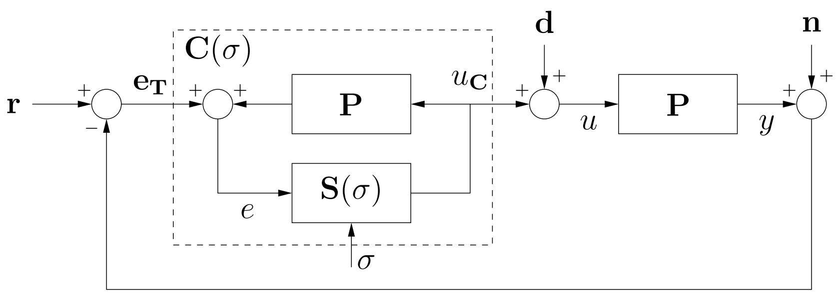


Fig. 5. Feedback connection between $\mathbf{P}$ and $\mathbf{C}(\sigma)$.

$$e = \mathbf{e_T} + \mathrm{H}_{\mathbf{P}} \circ u_{\mathbf{C}} + \epsilon_1, \tag{26}$$

where $\epsilon_1(t)$ is a signal that converges to zero exponentially fast and is due to nonzero initial conditions in the "copy" of the process inside the multi-controller. Also, for any $u_{\mathbf{C}}$,

$$\mathbf{e_T} = \mathbf{r} - \mathbf{n} - \mathrm{H}_{\mathbf{P}} \circ (u_{\mathbf{C}} + \mathbf{d}) + \epsilon_2, \tag{27}$$

where $\epsilon_2(t)$ also converges to zero exponentially fast and is due to nonzero initial conditions in the "real" process. From (26)–(27) we then conclude that

$$e = \mathbf{r} - \mathbf{n} - \mathrm{H}_{\mathbf{P}} \circ \mathbf{d} + \epsilon_1 + \epsilon_2. \tag{28}$$

This shows that e is independent of σ and will remain bounded, provided that $\mathbf{r}$, $\mathbf{n}$, and $\mathbf{d}$ are bounded.

Suppose now that we choose the reset matrices $\{\bar{R}(p,q) : p,q \in \mathcal{P}\}$ so that there exist symmetric, positive definite matrices $\{\bar{Q}_p \in \mathbb{R}^{n_S \times n_S} : p \in \mathcal{P}\}$ for which

$$\bar{Q}_p \bar{A}_p + \bar{A}_p' \bar{Q}_p < 0, \qquad \bar{R}(p,q)' \bar{Q}_p \bar{R}(p,q) \leq \bar{Q}_q, \qquad p,q \in \mathcal{P}. \tag{29}$$

Then, because of Lemma 1, $\mathbf{S}(\sigma)$ is exponentially stable, uniformly over $\mathcal{S}$ and its state x and output $u_{\mathbf{C}}$ remain bounded for every $\sigma \in \mathcal{S}$. Because of (27), $\mathbf{e}_{\mathbf{T}}$ is then also bounded, as well as all other signals. Moreover, if $\mathbf{r} = \mathbf{d} = \mathbf{n} = 0$ then e converges to zero exponentially fast, because of (28), and so does $u_{\mathbf{C}}$ and all the remaining signals.

It turns out that the overall closed-loop switched system (7)–(9), with the multi-controller built as above, is exponentially stable, uniformly over $\mathcal{S}$. This means that the properties derived above (namely, the boundedness of its state and convergence to zero in the absence of exogenous inputs) are robust with respect to small perturbations to the dynamics of the system (cf. Remark 2). In particular, these properties hold even if the "copy" of the process inside the multi-controller does not match exactly the real process. The fact that (7)–(9) is exponentially stable will be proved below for the general case.

Remark 4 *The choice of reset maps for which* (29) *holds is always possible. Either by enforcing complete reset, i.e.,* $\bar{R}(p,q) = 0$*, for all* $p,q \in \mathcal{P}$*, or by avoiding reset altogether through the choice of realizations for the* $\{\mathrm{S}_p : p \in \mathcal{P}\}$*, for which there is a common quadratic Lyapunov function. The latter is always possible as seen in Lemma 7 in the Appendix.*

4.2 General linear time-invariant process

The reader familiar with the Youla parameterization [15] probably recognized (24) as the general form of any controller that stabilizes the stable process $\mathrm{H}_{\mathbf{P}}$. It is well known that this formula can be generalized to multiple-input/multiple-output unstable linear time-invariant processes. We shall see shortly that the general formula is still amenable to the construction of multi-controllers adequate for stable switching.

Consider a multiple-input/multiple-output, possibly unstable, process transfer function $\mathrm{H}_{\mathbf{P}}$. To proceed we pick some controller transfer matrix K that stabilizes $\mathrm{H}_{\mathbf{P}}$. For example, one can take K to be one of the elements of $\mathcal{K}$. Because K stabilizes $\mathrm{H}_{\mathbf{P}}$, it is known [8] that there exist matrices A_E, B_E, C_E, D_E, F_E, and G_E (with appropriate dimensions) such that A_E is a stability

[8] Cf. Lemma 8 in the Appendix, which is a reformulation of results that can be found in [15,26,27].

matrix, and $\{A_E + D_E C_E, B_E, C_E\}$ and $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ are stabilizable and detectable realizations of $\mathrm{H}_{\mathbf{P}}$ and K, respectively.

Suppose that, for each $p \in \mathcal{P}$, we define

$$\mathrm{S}_p := \Big(- \mathrm{Y}_{\mathbf{C}} + \mathrm{X}_{\mathbf{C}} \mathrm{K}_p \Big) \Big(\mathrm{X}_{\mathbf{P}} + \mathrm{Y}_{\mathbf{P}} \mathrm{K}_p \Big)^{-1}, \tag{30}$$

where the four transfer matrices $\mathrm{X}_{\mathbf{C}}$, $\mathrm{Y}_{\mathbf{C}}$, $\mathrm{Y}_{\mathbf{P}}$, and $\mathrm{X}_{\mathbf{P}}$ are defined by

$$\begin{bmatrix} \mathrm{X}_{\mathbf{C}} & -\mathrm{Y}_{\mathbf{C}} \\ \mathrm{Y}_{\mathbf{P}} & \mathrm{X}_{\mathbf{P}} \end{bmatrix} := \begin{bmatrix} F_E \\ C_E \end{bmatrix} (sI - A_E)^{-1} \begin{bmatrix} B_E & -D_E \end{bmatrix} + \begin{bmatrix} I & -G_E \\ 0 & I \end{bmatrix}. \tag{31}$$

Using the fact that K_p stabilizes $\mathrm{H}_{\mathbf{P}}$ it is possible to establish that the poles of S_p must have negative real part. A straightforward derivation of this, using state-space methods, can be found in the appendix. This can also be proved using transfer function methods (cf. Remark 10 in the Appendix).

Solving (30) for K_p, we obtain

$$\mathrm{K}_p = \Big(\mathrm{X}_{\mathbf{C}} - \mathrm{S}_p \mathrm{Y}_{\mathbf{P}} \Big)^{-1} \Big(\mathrm{Y}_{\mathbf{C}} + \mathrm{S}_p \mathrm{X}_{\mathbf{P}} \Big). \tag{32}$$

Therefore the transfer function K_p from $\mathbf{e}_{\mathbf{T}}$ to $u_{\mathbf{C}}$ can be defined implicitly by the following system of equations

$$\begin{bmatrix} \bar{u} \\ e \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{\mathbf{C}} - I & -\mathrm{Y}_{\mathbf{C}} \\ \mathrm{Y}_{\mathbf{P}} & \mathrm{X}_{\mathbf{P}} \end{bmatrix} \circ \begin{bmatrix} u_{\mathbf{C}} \\ \mathbf{e}_{\mathbf{T}} \end{bmatrix}, \qquad v = \mathrm{S}_p \circ e, \qquad u_{\mathbf{C}} = v - \bar{u}. \tag{33}$$

This is because we conclude from (33) that

$$\begin{aligned} u_{\mathbf{C}} = v - \bar{u} &= \mathrm{S}_p \circ \Big(\mathrm{Y}_{\mathbf{P}} \circ u_{\mathbf{C}} + \mathrm{X}_{\mathbf{P}} \circ \mathbf{e}_{\mathbf{T}} \Big) - \big(\mathrm{X}_{\mathbf{C}} \circ u_{\mathbf{C}} - u_{\mathbf{C}} - \mathrm{Y}_{\mathbf{C}} \circ \mathbf{e}_{\mathbf{T}} \big) \\ &= -(\mathrm{X}_{\mathbf{C}} - I - \mathrm{S}_p \mathrm{Y}_{\mathbf{P}}) \circ u_{\mathbf{C}} + (\mathrm{Y}_{\mathbf{C}} + \mathrm{S}_p \mathrm{X}_{\mathbf{P}}) \circ \mathbf{e}_{\mathbf{T}}, \end{aligned}$$

and therefore

$$(\mathrm{X}_{\mathbf{C}} - \mathrm{S}_p \mathrm{Y}_{\mathbf{P}}) \circ u_{\mathbf{C}} = (\mathrm{Y}_{\mathbf{C}} + \mathrm{S}_p \mathrm{X}_{\mathbf{P}}) \circ \mathbf{e}_{\mathbf{T}}.$$

The transfer function in (32) follows directly. Pick now a realization $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ for S_p, because of (31) the system of equations (33) can be realized as

$$\dot{x} = \bar{A}_p x + \bar{B}_p e, \qquad v = \bar{C}_p x + \bar{D}_p e, \tag{34}$$

$$\dot{x}_E = A_E x_E + B_E u_{\mathbf{C}} - D_E \mathbf{e}_{\mathbf{T}}, \qquad e = C_E x_E + \mathbf{e}_{\mathbf{T}}, \tag{35}$$

$$u_{\mathbf{C}} = -F_E x_E + G_E \mathbf{e}_{\mathbf{T}} + v, \tag{36}$$

which must then realize K_p. It is important to note that only (34) changes from controller to controller. We shall use (33)—or more precisely, its state space version (34)–(36)—to guide us in constructing the multi-controller. Indeed, (33) will replace the equation (25) used in Section 4.1 for the same effect. The following steps are required:

(1) Pick stabilizable and detectable n_S-dimensional realizations $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ for each S_p, $p \in \mathcal{P}$, defined by (30).

(2) Define $\mathbf{S}(\sigma)$ to be the system with impulse effect defined by

$$\dot{x} = \bar{A}_\sigma x + \bar{B}_\sigma e, \qquad\qquad v = \bar{C}_\sigma x + \bar{D}_\sigma e,$$

on intervals where σ is constant and by

$$x(t) = \bar{R}\Big(\sigma(t), \sigma(t^-)\Big)\, x(t^-),$$

at each switching time t of σ. The reset matrices $\{\bar{R}(p,q) : p, q \in \mathcal{P}\}$ should be chosen so that there exist symmetric, positive definite matrices $\{\bar{Q}_p \in \mathbb{R}^{n_S \times n_S} : p \in \mathcal{P}\}$, such that

$$\bar{Q}_p \bar{A}_p + \bar{A}_p' \bar{Q}_p < 0, \qquad \bar{R}(p,q)' \bar{Q}_p \bar{R}(p,q) \leq \bar{Q}_q, \qquad p, q \in \mathcal{P}. \tag{37}$$

Because of Lemma 1, $\mathbf{S}(\sigma)$ is exponentially stable, uniformly over $\mathcal{S}$. Also here, the choice of reset maps for which (37) holds is always possible (cf. Remark 4).

(3) Realize the switching controller as

$$\begin{aligned} \dot{x}_E &= A_E x_E + B_E u_{\mathbf{C}} - D_E \mathbf{e_T}, \qquad\qquad e = C_E x_E + \mathbf{e_T}, \\ u_{\mathbf{C}} &= -F_E x_E + G_E \mathbf{e_T} + v, \end{aligned}$$

where e and v are the input and output of $\mathbf{S}(\sigma)$, respectively. This corresponds to the multi-controller in (5)–(6) with

$$F_p := \begin{bmatrix} A_E - B_E F_E + B_E \bar{D}_p C_E & B_E \bar{C}_p \\ \bar{B}_p C_E & \bar{A}_p \end{bmatrix}, \qquad G_p := \begin{bmatrix} -D_E + B_E(\bar{D}_p + G_E) \\ \bar{B}_p \end{bmatrix}, \tag{38}$$

$$H_p := \begin{bmatrix} -F_E + \bar{D}_p C_E & \bar{C}_p \end{bmatrix}, \qquad J_p := \bar{D}_p + G_E, \tag{39}$$

and

$$R_{\mathbf{C}}(p,q) := \begin{bmatrix} I & 0 \\ 0 & \bar{R}(p,q) \end{bmatrix}. \tag{40}$$

As in the case of single-input/single-output stable processes, it is possible to show that the signal e is independent of σ and remains bounded. However, instead of proceeding along this line of reasoning, we shall show directly that the overall closed-loop switched system (7)–(9), with the multi-controller built as above, is exponentially stable, uniformly over $\mathcal{S}$.

Theorem 5 *There exist symmetric matrices $\{Q_p : p \in \mathcal{P}\}$ for which* (18)–(19) *hold with $\{A_p, R(p,q) : p, q \in \mathcal{P}\}$ as in* (8) *and* (10)*, where the process* *realization* $\{A, B, C\}$ *is given by*

$$A := A_E + D_E C_E, \qquad B := B_E, \qquad C := C_E, \tag{41}$$

the controller realizations $\{F_p, G_p, H_p, J_p : p \in \mathcal{P}\}$ *are given by* (38)–(39), *and the controller reset matrices* $\{R_{\mathbf{C}}(p,q) : p, q \in \mathcal{P}\}$ *are given by* (40). *The closed-loop system with impulse effects* (7)–(9) *is therefore exponentially stable, uniformly over* $\mathcal{S}$.

Before proving Theorem 5, it should be noted that, in general, the realizations given by (38)–(39) are not minimal. However, denoting by n_K the McMillan degree of K, by n_H the McMillan degree of $\mathrm{H}_{\mathbf{P}}$, and by $n_{\mathcal{K}}$ the McMillan degree of the transfer matrix in $\mathcal{K}$ with largest McMillan degree, the size of A_E need not be larger than $n_H + n_K$ (cf. Lemma 8) and therefore the dimension of the state of the realizations (38)–(39) need not be larger than $2(n_H + n_K) + n_{\mathcal{K}}$ *no matter what the number of controllers in* $\mathcal{K}$ *is*. When K is chosen to have the structure of an observer with state feedback , i.e., when $\mathrm{H}_{\mathbf{P}}$ and K have realizations $\{A, B, C\}$ and $\{A + HC - BF, H, F\}$, respectively, the size of the matrix A_E need not be larger than n_H (cf. Remark 9) and therefore the dimension of the state of the realizations (38)–(39) can be reduced to $2n_H + n_{\mathcal{K}}$.

Proof of Theorem 5. Replacing (38)–(39) and (41) in (8), one obtains for the closed-loop system:

$$A_p = \begin{bmatrix} A_E + D_E C_E - B_E(\bar{D}_p + G_E)C_E & -B_E F_E + B_E \bar{D}_p C_E & B_E \bar{C}_p \\ D_E C_E - B_E(\bar{D}_p + G_E)C_E & A_E - B_E F_E + B_E \bar{D}_p C_E & B_E \bar{C}_p \\ -\bar{B}_p C_E & \bar{B}_p C_E & \bar{A}_p \end{bmatrix}, \qquad p \in \mathcal{P}.$$

Defining $T := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{bmatrix}$, one further concludes that

$$T A_p T^{-1} = \begin{bmatrix} A_E + D_E C_E - B_E F_E - B_E G_E C_E & B_E \bar{C}_p & -B_E F_E + B_E \bar{D}_p C_E \\ 0 & \bar{A}_p & \bar{B}_p C_E \\ 0 & 0 & A_E \end{bmatrix}. \tag{42}$$

Here, we used the fact that $T^{-1} = \begin{bmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{bmatrix}$. Since K stabilizes $\mathrm{H}_{\mathbf{P}}$ and $\{A_E + D_E C_E, B_E, C_E\}$ and $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ are stabilizable and detectable realizations of $\mathrm{H}_{\mathbf{P}}$ and K, respectively, the matrix

$$\bar{A}_E := \begin{bmatrix} A_E + D_E C_E - B_E G_E C_E & B_E F_E \\ -(D_E - B_E G_E)C_E & A_E - B_E F_E \end{bmatrix} \tag{43}$$

is asymptotically stable (cf. right-hand side of (43) against (4)). Therefore,

$$T_E \bar{A}_E T_E^{-1} = \begin{bmatrix} A_E + D_E C_E - B_E F_E - B_E G_E C_E & B_E F_E \\ 0 & A_E \end{bmatrix},$$

with $T_E := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$, $T_E^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ is also asymptotically stable. The matrices $A_E + D_E C_E - B_E F_E - B_E G_E C_E$ and A_E must then be asymptotically stable and so there exist positive definite symmetric matrices Q_1, Q_2 such that

$$\begin{aligned} Q_1(A_E+&D_EC_E - B_EF_E - B_EG_EC_E) \\ &+ (A_E + D_EC_E - B_EF_E - B_EG_EC_E)'Q_1 = -I \qquad (44) \\ Q_2A_E+&A_E'Q_2 = -I. \qquad (45) \end{aligned}$$

Moreover, because of (37), each

$$P_p := -\bar{Q}_p\bar{A}_p - \bar{A}_p'\bar{Q}_p, \qquad p \in \mathcal{P},$$

is positive definite. Therefore there must exist a positive constant ϵ, sufficiently small, such that

$$P_p - \epsilon\bar{Q}_p\bar{B}_pC_EC_E'\bar{B}_p'\bar{Q}_p > 0, \qquad \forall p \in \mathcal{P},$$

which guarantees that each

$$R_p := \epsilon \begin{bmatrix} P_p & -\bar{Q}_p\bar{B}_pC_E \\ -C_E'\bar{B}_p'\bar{Q}_p & \epsilon^{-1}I \end{bmatrix}, \qquad p \in \mathcal{P}, \qquad (46)$$

is also positive definite (cf. [3, Section 2.1]). Let now

$$Q_p := T' \begin{bmatrix} \epsilon_1Q_1 & 0 & 0 \\ 0 & \epsilon\bar{Q}_p & 0 \\ 0 & 0 & Q_2 \end{bmatrix} T, \qquad (47)$$

with

$$\epsilon_1 := \frac{1}{2}\left(\max_{p\in\mathcal{P}} \|Q_1S_pR_p^{-1}S_p'Q_1\|\right)^{-1}, \qquad (48)$$

where, for each $p \in \mathcal{P}$,

$$S_p := \begin{bmatrix} B_E\bar{C}_p & -B_EF_E+B_E\bar{D}_pC_E \end{bmatrix}. \qquad (49)$$

From (42), (44)–(45), (46), (47), and (49) one concludes that

$$Q_pA_p + A_p'\,Q_p = -\epsilon_1T' \begin{bmatrix} I & -Q_1S_p \\ -S_p'Q_1 & \epsilon_1^{-1}R_p \end{bmatrix} T, \qquad p \in \mathcal{P}. \qquad (50)$$

But, because of (48), $I - \epsilon_1Q_1S_pR_p^{-1}S_p'Q_1 > 0$ for each $p \in \mathcal{P}$, thus

$$\begin{bmatrix} I & -Q_1S_p \\ -S_p'Q_1 & \epsilon_1^{-1}R_p \end{bmatrix} > 0, \qquad p \in \mathcal{P}$$

(cf. [3, Section 2.1]). From this and (50) one concludes that (18) holds.

The inequality (19) is a straightforward consequence of the definitions of the Q_p in (47) and the $R(p,q)$ in (10), (40). Indeed, from these definitions one concludes that

$$R(p,q)'Q_pR(p,q) - Q_q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon\left(\bar{R}(p,q)'\bar{Q}_p\bar{R}(p,q) - \bar{Q}_q\right) \end{bmatrix}, \qquad p, q \in \mathcal{P}.$$

The matrix on the right-hand side is negative semi-definite because of (37). □

Remark 6 *In Theorem 5, we prove the existence of the matrices* $\{Q_p : p \in \mathcal{P}\}$ *needed to apply Lemma 1 for a specific realization* (41) *of the process transfer matrix* $\mathrm{H}_{\mathbf{P}}$*. This may not be the "actual" realization of the process and not even similar to it (as* (41) *may not minimal). However, this is irrelevant as far as the exponential stability of the switched system is concerned because (i) asymptotically stable modes of the process that are not observable do not affect the switched controller and (ii) only the controllable modes of the process can be excited by the multi-controller.*

5 Example

In this section we briefly illustrate how to utilize the results presented above in a design problem. We consider here the control of the roll angle of an aircraft. The following process model was taken from [28, p. 381]:

$$\mathrm{H}_{\mathbf{P}}(s) = \frac{-1000}{s(s+.875)(s+50)}.$$

Ideally, one would like to design a controller that is both fast and has good measurement noise rejection properties. Clearly this is not possible, as increasing the bandwidth of the closed-loop system will also make the system more sensitive to measurement noise. We opt then to design two distinct controllers: Controller K_1 has low closed-loop bandwidth and is therefore not very sensitive to noise but exhibits a slow response. Controller K_2 has high bandwidth and is therefore fast but very sensitive to noise. Both controllers were designed using LQG/LQR. We computed the regulator gains by minimizing the cost

$$J_{\text{reg}} := \int_0^\infty y^2(\tau) + \dot{y}^2(\tau) + \rho u^2(\tau) d\tau$$

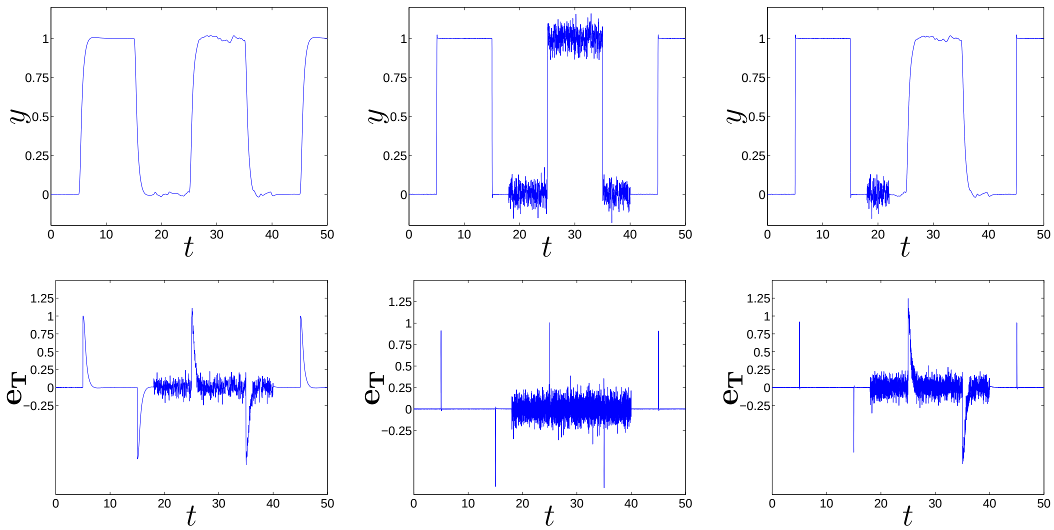
where ρ was chosen equal to 100 and .1 for K_1 and K_2, respectively. These choices of ρ resulted in K_2 exhibiting a much fast response than K_1. The design of the optimal LQG gain was done assuming that the input disturbance $\mathbf{d}$ and the measurement noise $\mathbf{n}$ were uncorrelated white noise processes with

$$\mathrm{E}[\mathbf{d}(t)\mathbf{d}(\tau)] = \delta(t-\tau), \qquad \mathrm{E}[\mathbf{n}(t)\mathbf{n}(\tau)] = \mu\delta(t-\tau),$$

where μ was chosen equal to 10^{-1} and 10^{-10} for K_1 and K_2, respectively. These choices of μ resulted in K_1 exhibiting much better noise rejection properties than K_2. The controller transfer functions obtained were:

$$\mathrm{K}_1 \approx \frac{-6.694(s+.9446)(s+50.01)}{(s^2+13.23s+9.453^2)(s+50.05)}, \qquad \mathrm{K}_2 \approx \frac{-2187^2(s+.9977)(s+66.28)}{(s^2+467.2s+486.2^2)(s+507)}.$$

The two left plots in Figure 6 show the closed-loop response of controllers K_1 and K_2 to a square reference. Large measurement noise was injected into the system for $t \in [18, 40]$. By design, controller K_1 exhibits a faster response but is more sensitive to measurement noise.



Controller K_1 Controller K_2 Switched controller

Fig. 6. Closed-loop response of controllers K_1, K_2, and the switched multi-controller to a square reference r. Large measurement noise $\mathbf{n}$ was injected into the system in the interval $t \in [18, 40]$. The top plots show the output y and the bottom plots the tracking error $\mathbf{e_T} := \mathbf{r} - y - \mathbf{n}$. For the switched controller, K_1 was used in the interval $t \in [22, 42]$ and K_2 in the remaining time.

To design the multi-controller for $\mathcal{K} := \{\mathrm{K}_1, \mathrm{K}_2\}$ we followed the procedure given in Section 4.2: We started by selecting matrices A_E, B_E, C_E, D_E, F_E, and G_E such that A_E is a stability matrix, and $\{A_E + D_E C_E, B_E, C_E\}$ and $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ are stabilizable and detectable realizations of $\mathrm{H}_\mathbf{P}$ and $\mathrm{K} := \mathrm{K}_1$, respectively. Since K has the structure of an observer with state feedback, we used the formulas in Remark 9 for these matrices. The corresponding transfer matrices $\{\mathrm{S}_1, \mathrm{S}_2\}$ were then computed using (30):

$$\mathrm{S}_1 = 0, \qquad \mathrm{S}_2 = \tfrac{2187^2(s+.9995)(s^2+13.26s+9.319^2)(s+50.05)(s+66.24)}{(s+1)(s^2+93.94s+56.23^2)(s^2+465.1s+463.7^2)(s+465.1)}.$$

The fact that $\mathrm{S}_1 = 0$ is a consequence of having used $\mathrm{K} := \mathrm{K}_1$ (cf. Remark 10). We then picked a minimal realization $\{\bar{A}_2, \bar{B}_2, \bar{C}_2\}$ for S_2 and the trivial realization $\{\bar{A}_2, 0, \bar{C}_2\}$ for S_1. Since both realization share the same stable $\bar{A}_2$ matrix, (37) holds with $\bar{Q}_1 = \bar{Q}_2$ and $\bar{R}(1,2) = \bar{R}(2,1) = I$. As mentioned before, it would have been possible to choose realization for S_1 and S_2 with this property even if S_1 was nontrivial. The desired controller realizations are then given by (38)–(39), and the controller reset matrices are simply the identity (i.e., no reset is used). These guarantee that the switched closed-loop system is exponentially stable, uniformly over $\mathcal{S}$.

The rightmost plot in Figure 6 shows the closed-loop response of the switched controller. In this figure, controller K_2 was used until time $t = 22$ (shortly after the measurement noise increased). At that point there was a switch to controller K_1, resulting in significant noise rejection. Controller K_1 was used until time $t = 42$ (shortly after the measurement noise decreased back to the original level). The construction of a logic that actually commands the switching between controllers is beyond the scope of this paper. The contribution here is the implementation of the multi-controller so that we have stability *regardless of the switching signal* σ. Once stability is guaranteed, one can use very simpleminded algorithms to decide how to switch between the controllers. For example, one could use controller K_2 only when there is low high-frequency content in the tracking error $\mathbf{e_T}$.

6 Conclusions

In the control of complex systems, conflicting requirements often make a single linear time-invariant controller unsuitable. One can then be tempted to design several controllers, each suitable for a specific operating condition, and switch among them to achieve the best possible performance. Unfortunately, it is well known that the transients caused by switching may cause instability. We showed here that instability can be avoided by suitable choice of the realizations for the controllers.

An important question for future research is the design of logics that orchestrate the switching among controllers to improve performance. The results in this paper greatly simplify the design of such logics since stability of the switched system is no longer an issue. Another question that needs to be investigated is the simultaneous switching of process and controller. In particular, suppose that the process to be controlled switches in an unpredictable fashion and that we would like to switch controllers to keep the closed-loop system stable. Can we choose realizations for the controllers so that the process/controller switched system is stable? An affirmative answer to this question would have a profound impact both in gain-scheduling and in multiple-model supervisory control (cf. [29–31]).

A Appendix

A.1 Realizations for stable transfer matrices

This section addresses a simpler problem than the one formulated before. Consider a finite family of *asymptotically stable* transfer matrices $\mathcal{A} = \{\mathrm{S}_p : p \in \mathcal{P}\}$. It is shown below how to compute stabilizable and detectable n-dimensional realizations $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ for each $\mathrm{S}_p \in \mathcal{A}$ such that

$$Q\bar{A}_p + \bar{A}_p'Q < 0, \qquad p \in \mathcal{P}, \tag{A.1}$$

for some symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$. With such a matrix it is then possible to construct a common Lyapunov function $V(z) = z'Qz$ for the family of linear time-invariant systems $\{\dot{z} = \bar{A}_p z : p \in \mathcal{P}\}$.

Let n be the McMillan degree of the transfer matrix in $\mathcal{A}$ with largest McMillan degree and, for each $p \in \mathcal{P}$, let $\{\tilde{A}_p, \tilde{B}_p, \tilde{C}_p, \tilde{D}_p\}$ be any n-dimensional realization of S_p, with $\tilde{A}_p$ asymptotically stable. Because of the asymptotic stability of each $\tilde{A}_p$, $p \in \mathcal{P}$ the family of Lyapunov equations

$$Q_p\tilde{A}_p + \tilde{A}_p'Q_p = -I, \qquad p \in \mathcal{P} \tag{A.2}$$

must have symmetric positive definite solutions Q_p, which can be written as $Q_p = S_p'S_p$ with S_p nonsingular. For a given positive definite matrix $Q = S'S \in \mathbb{R}^{n\times n}$ with S nonsingular, let

$$\bar{A}_p := S^{-1}S_p\tilde{A}_pS_p^{-1}S, \quad \bar{B}_p := S^{-1}S_p\tilde{B}_p, \quad \bar{C}_p := \tilde{C}_pS_p^{-1}S, \quad \bar{D}_p := \tilde{D}_p, \tag{A.3}$$

Since $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ is obtained from $\{\tilde{A}_p, \tilde{B}_p, \tilde{C}_p, \tilde{D}_p\}$ by a similarity transformation, $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ is also a realization of S_p. Moreover, from (A.2) and (A.3) we conclude that

$$(S^{-1}S_p)'\Big(Q\bar{A}_p + \bar{A}_p'Q\Big)S^{-1}S_p = -I.$$

Left and right multiplication of the above equality by $(S_p^{-1}S)'$ and $S_p^{-1}S$, respectively, yields

$$Q\bar{A}_p + \bar{A}_p'Q = -(S_p^{-1}S)'S_p^{-1}S < 0$$

and therefore one concludes that (A.1) holds. The following was proved:

Lemma 7 *Given any finite family of asymptotically stable transfer matrices* $\mathcal{A} = \{\mathrm{S}_p : p \in \mathcal{P}\}$ *with McMillan degree no larger than* n *and any symmetric positive definite* $n \times n$ *matrix* Q*, there exist stabilizable and detectable* n*-dimensional realizations* $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$ *for each* $\mathrm{S}_p \in \mathcal{A}$ *such that* (A.1) *holds.*

Lemma 8 *Given two transfer matrices* N *and* K*, with* N *strictly proper, such that* K *stabilizes* N*, there exist matrices* A_E*,* B_E*,* C_E*,* D_E*,* F_E*, and* G_E *(with appropriate dimensions) such that* A_E *is a stability matrix, and* $\{A_E + D_E C_E, B_E, C_E\}$ *and* $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ *are stabilizable and detectable realizations of* $\mathrm{H}_{\mathbf{P}}$ *and* K*, respectively.*

Proof of Lemma 8. Let $\{A, B, C\}$ and $\{F, G, H, J\}$ be minimal realizations of $\mathrm{H}_{\mathbf{P}}$ and K, respectively, and X, Y matrices such that $A+XC$ and $F+YH$ are asymptotically stable. Defining

$$A_E := \begin{bmatrix} A+XC & 0 \\ 0 & F+YH \end{bmatrix}, \qquad B_E := \begin{bmatrix} B \\ -Y \end{bmatrix}, \qquad D_E := \begin{bmatrix} -X \\ -G-YJ \end{bmatrix},$$
$$C_E := \begin{bmatrix} C & 0 \end{bmatrix}, \qquad F_E := \begin{bmatrix} 0 & -H \end{bmatrix}, \qquad G_E := J,$$

the matrix A_E is asymptotically stable and

$$C_E(sI - A_E - D_E C_E)^{-1} B_E =$$
$$= \begin{bmatrix} C & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A & 0 \\ -GC-YJC & F+YH \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ -Y \end{bmatrix} = \mathrm{H}_{\mathbf{P}}(s),$$
$$F_E(sI - A_E + B_E F_E)^{-1}(D_E - B_E G_E) + G_E =$$
$$= \begin{bmatrix} 0 & -H \end{bmatrix} \left(sI - \begin{bmatrix} A+XC & BH \\ 0 & F \end{bmatrix} \right)^{-1} \begin{bmatrix} -X-BJ \\ -G \end{bmatrix} + J = \mathrm{K}(s).$$

Detectability of $\{C_E, A_E + D_E C_E\}$ and $\{F_E, A_E - B_E F_E\}$ is guaranteed by the fact that both $A_E + D_E C_E$ and $A_E - B_E F_E$ are an output injection away from A_E which is a stability matrix. Stabilizability of $\{A_E + D_E C_E, B_E\}$ and $\{A_E - B_E F_E, D_E - B_E G_E\}$ is guaranteed by the fact that both $A_E + D_E C_E$ and $A_E - B_E F_E$ are a state feedback away from

$$\begin{bmatrix} A-BJC & BH \\ -GC & F \end{bmatrix},$$

which is a stability matrix since K stabilizes $\mathrm{H}_{\mathbf{P}}$. □

Remark 9 *When* K *is chosen to have the structure of an observer with state feedback, i.e., when* $\mathrm{H}_{\mathbf{P}}$ *and* K *have realizations* $\{A, B, C\}$ *and* $\{A + HC - BF, -H, F\}$*, respectively, one can simply pick* $A_E = A + HC$*,* $B_E = B$*,* $C_E = C$*,* $D_E = -H$*,* $F_E = F$*, and* $G_E = 0$*.*

Verification of the stability of the S_p**,** $p \in \mathcal{P}$**.** Straightforward algebra shows that the transfer function on the right-hand side of (30) is equal to the transfer function from e to v defined by the system of equations

$$\begin{bmatrix} v \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{\mathbf{C}} & -\mathrm{Y}_{\mathbf{C}} \\ \mathrm{Y}_{\mathbf{P}} & \mathrm{X}_{\mathbf{P}} - I \end{bmatrix} \circ \begin{bmatrix} \bar{u} \\ e - \bar{y} \end{bmatrix}, \qquad \bar{u} = \mathrm{K}_p \circ (e - \bar{y}). \tag{A.4}$$

Now, because of (31),

$$\left\{ A_E, \begin{bmatrix} B_E & -D_E \end{bmatrix}, \begin{bmatrix} F_E \\ C_E \end{bmatrix}, \begin{bmatrix} I & -G_E \\ 0 & 0 \end{bmatrix} \right\}$$

is a realization for $\left[\begin{smallmatrix} \mathrm{X}_{\mathbf{C}} & -\mathrm{Y}_{\mathbf{C}} \\ \mathrm{Y}_{\mathbf{P}} & \mathrm{X}_{\mathbf{P}} - I \end{smallmatrix}\right]$. Thus, picking any minimal realization $\{\hat{A}_p, \hat{B}_p, \hat{C}_p, \hat{D}_p\}$ of K_p, the system (A.4) can be realized as

$$\begin{aligned} \dot{x}_E &= A_E x_E + B_E \bar{u} - D_E(e - \bar{y}), & \bar{y} &= C_E x_E, \\ \dot{\hat{x}} &= \hat{A}_p \hat{x} + \hat{B}_p(e - \bar{y}), & \bar{u} &= \hat{C}_p \hat{x} + \hat{D}_p(e - \bar{y}), \\ v &= F_E x_E + \bar{u} - G_E(e - \bar{y}). \end{aligned}$$

Therefore, the transfer function from e to v in (A.4) (and therefore S_p) can be realized by $\{\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p\}$, with

$$\bar{A}_p := \begin{bmatrix} A_E + D_E C_E - B_E \hat{D}_p C_E & B_E \hat{C}_p \\ -\hat{B}_p C_E & \hat{A}_p \end{bmatrix}, \tag{A.5}$$

and $\bar{B}_p$, $\bar{C}_p$, $\bar{D}_p$ appropriately defined. Since K_p stabilizes $\mathrm{H}_{\mathbf{P}}$ and $\{A_E + D_E C_E, B_E, C_E\}$ is a stabilizable and detectable realization of $\mathrm{H}_{\mathbf{P}}$, $\bar{A}_p$ must be asymptotically stable (cf. $\bar{A}_p$ in (A.5) against (4)). Thus, for each $p \in \mathcal{P}$, the poles of S_p must also have negative real part.

Remark 10 *Denoting by* RH_∞ *the ring of transfer matrices whose entries are proper, stable rational functions with real coefficients, the transfer matrices* $\mathrm{X}_{\mathbf{P}}, \mathrm{Y}_{\mathbf{P}}, \mathrm{Y}_{\mathbf{C}}, \mathrm{X}_{\mathbf{C}}$ *defined in* (31) *form a simultaneous right-coprime factorization of* $\mathrm{H}_{\mathbf{P}}$ *and* K *in the sense that* $\mathrm{X}_{\mathbf{P}}$ *and* $\mathrm{X}_{\mathbf{C}}$ *have causal inverse,* $\left[\begin{smallmatrix} \mathrm{X}_{\mathbf{C}} & -\mathrm{Y}_{\mathbf{C}} \\ \mathrm{Y}_{\mathbf{P}} & \mathrm{X}_{\mathbf{P}} \end{smallmatrix}\right]$ *is a unit in* RH_∞*, and* $\mathrm{H}_{\mathbf{P}} = \mathrm{X}_{\mathbf{P}}^{-1}\mathrm{Y}_{\mathbf{P}}$ *and* $\mathrm{K} = \mathrm{X}_{\mathbf{C}}^{-1}\mathrm{Y}_{\mathbf{C}}$*. Thus, the existence of the family of stable transfer matrices* $\{\mathrm{S}_p : p \in \mathcal{P}\} \subset \mathrm{RH}_\infty$ *such that* (32) *holds is not surprising in light of the Youla parameterization of all controllers that stabilize* $\mathrm{H}_{\mathbf{P}}$*, given by [15]. Note also that since* $\mathrm{K} = \mathrm{X}_{\mathbf{C}}^{-1}\mathrm{Y}_{\mathbf{C}}$*, if one chooses* $\mathrm{K} = \mathrm{K}_{p_0}$ *for some* $p_0 \in \mathcal{P}$*, then the corresponding transfer matrix* S_{p_0} *given by* (30) *with* $p = p_0$ *is equal to 0.*

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