Networked Control System Protocols
Modeling & Analysis
using Stochastic Impulsive Systems

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Talk outline

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Examples

- feedback over shared communication network
- estimation using remote sensor

Analysis tools

- Stochastic Hybrid Systems driven by renewal processes
- Lyapunov-based analysis of Stochastic Hybrid Systems

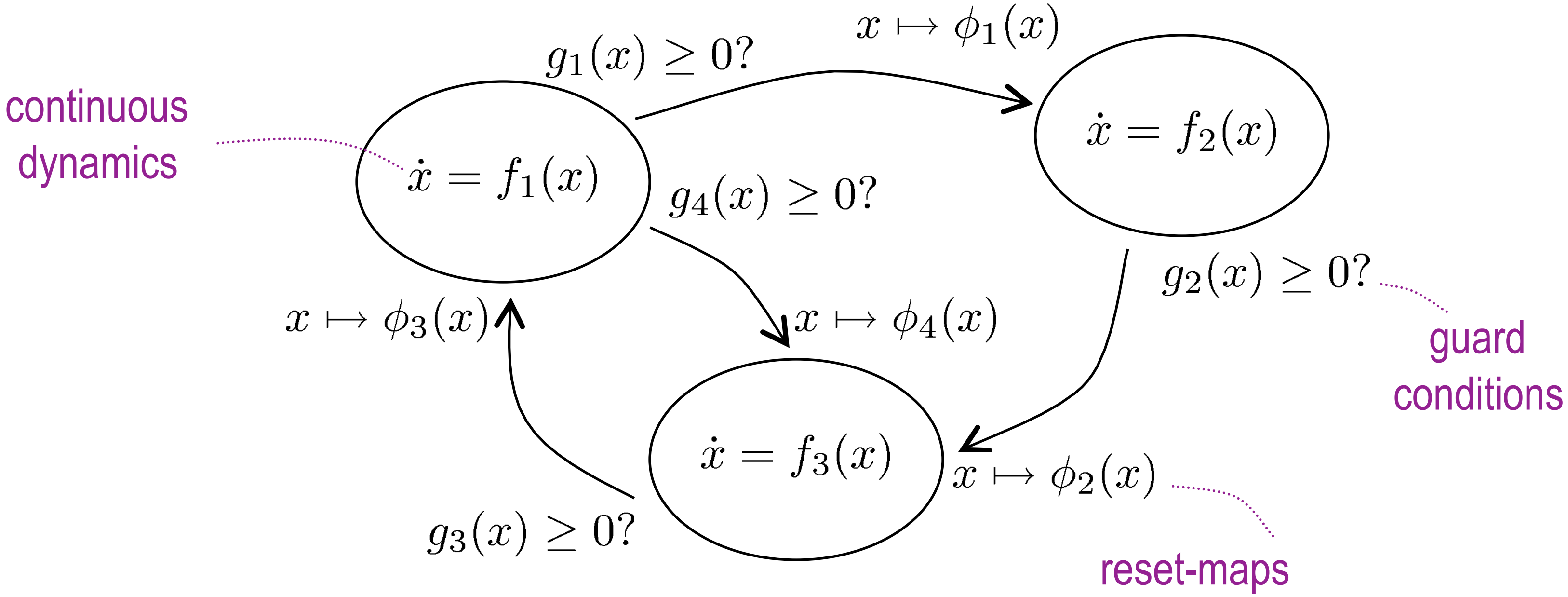
(ex) students: D. Antunes (IST), Y. Xu (Advertising.com)

collaborators: C. Silvestre (IST)

acknowledgements: NSF, AFOSR (STTR program)

disclaimer: This is an overview, technical details in papers referenced in bottom right corner… http://www.ece.ucsb.edu/~hespanha

Deterministic Hybrid Systems



continuous dynamics

$\dot{x} = f_1(x)$

$\dot{x} = f_2(x)$

$\dot{x} = f_3(x)$

$g_1(x) \geq 0?$

$x \mapsto \phi_1(x)$

$g_4(x) \geq 0?$

$x \mapsto \phi_4(x)$

$g_2(x) \geq 0?$

guard conditions

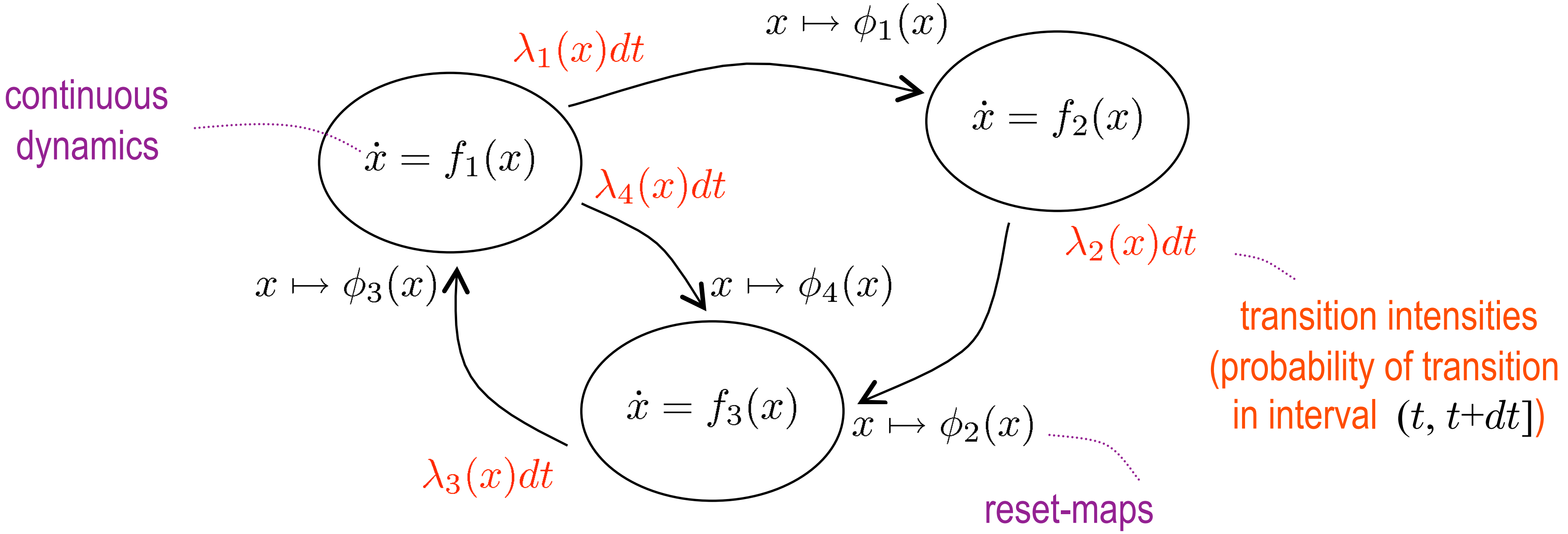
$x \mapsto \phi_2(x)$

reset-maps

$g_3(x) \geq 0?$

$x \mapsto \phi_3(x)$

Stochastic Hybrid Systems



continuous dynamics

$\dot{x} = f_1(x)$

$\dot{x} = f_2(x)$

$\dot{x} = f_3(x)$

$\lambda_1(x)dt$

$x \mapsto \phi_1(x)$

$\lambda_4(x)dt$

$x \mapsto \phi_4(x)$

$\lambda_2(x)dt$

$x \mapsto \phi_2(x)$

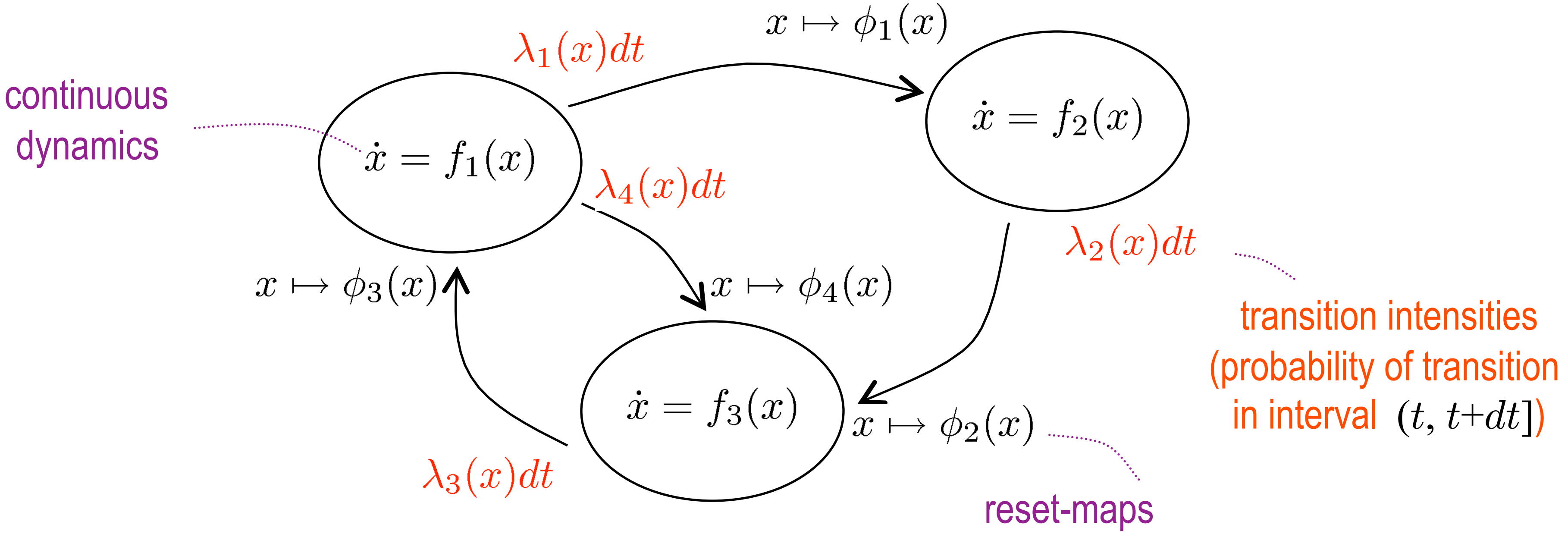
$\lambda_3(x)dt$

$x \mapsto \phi_3(x)$

transition intensities
(probability of transition
in interval $(t, t+dt])$

reset-maps

Stochastic Hybrid Systems



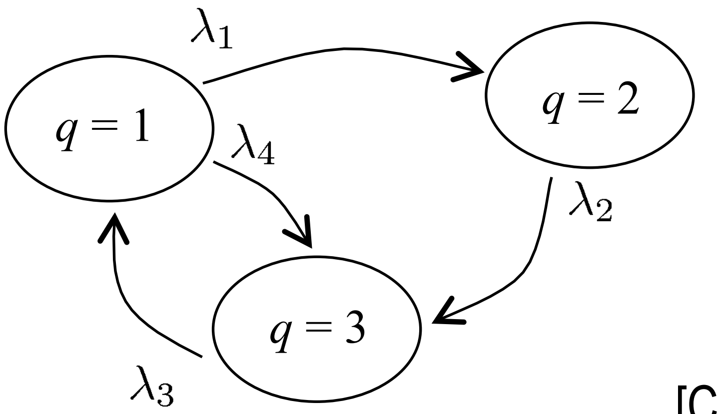
continuous dynamics

transition intensities (probability of transition in interval $(t, t+dt]$)

reset-maps

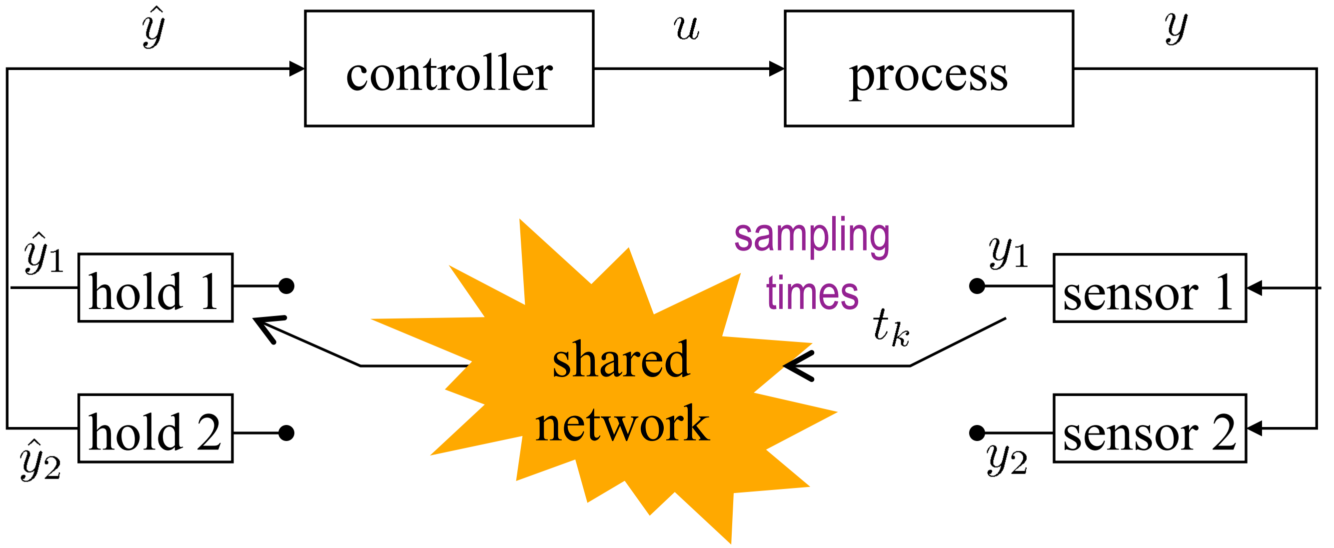
Special case: When all λ_ℓ are constant

$\Rightarrow$ $x(t)$ is a Markov process &
times between jumps are exponentially distributed



closely related to the so called
Markovian Jump Systems
[Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]

Example I: Networked Control System



process:
$$\dot{x}_P = A_P x_P + C_P u$$
$$y = C_P x_P + D_P u$$

controller:
$$\dot{x}_C = A_C x_C + C_C \hat{y}$$
$$\hat{y} = C_C x_C + D_C \hat{y}$$

round-robin network access:

$$\dot{\hat{y}} = 0$$

hold

$$\hat{y}(t_k) = \begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{cases} \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix} & k \text{ odd} \\ \begin{bmatrix} \hat{y}_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} & k \text{ even} \end{cases}$$

Example I: Networked Control System

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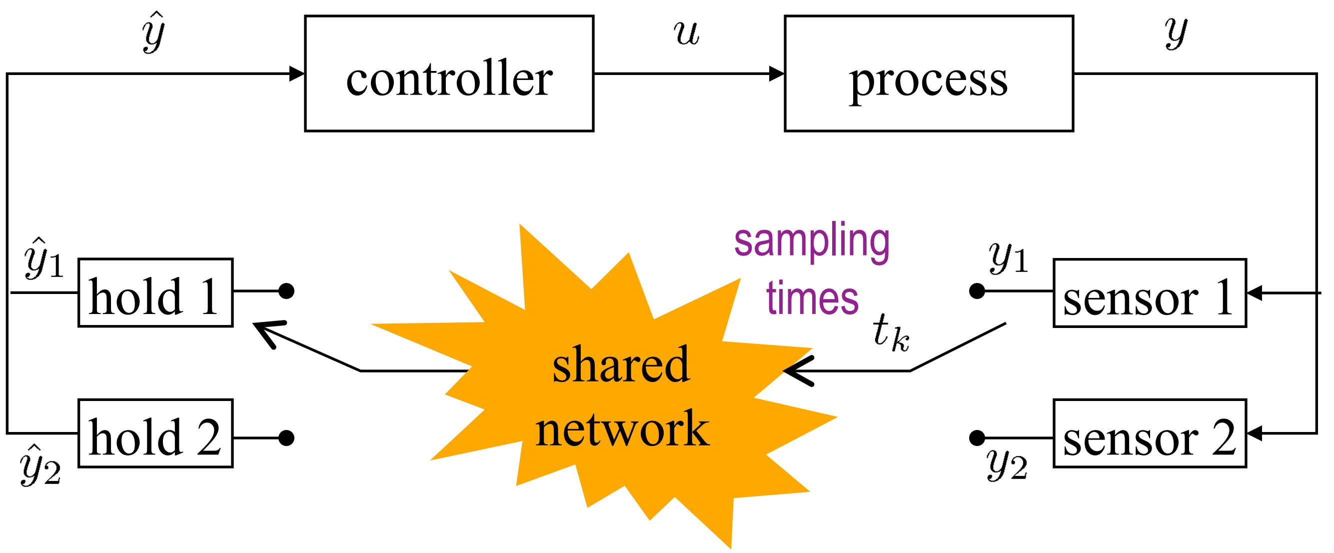


process: $\dot{x}_P = A_P x_P + C_P u$

$y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$

$\hat{y} = C_C x_C + D_C \hat{y}$

What if the network is not available at a sample time t_k ?

1st wait until network becomes available

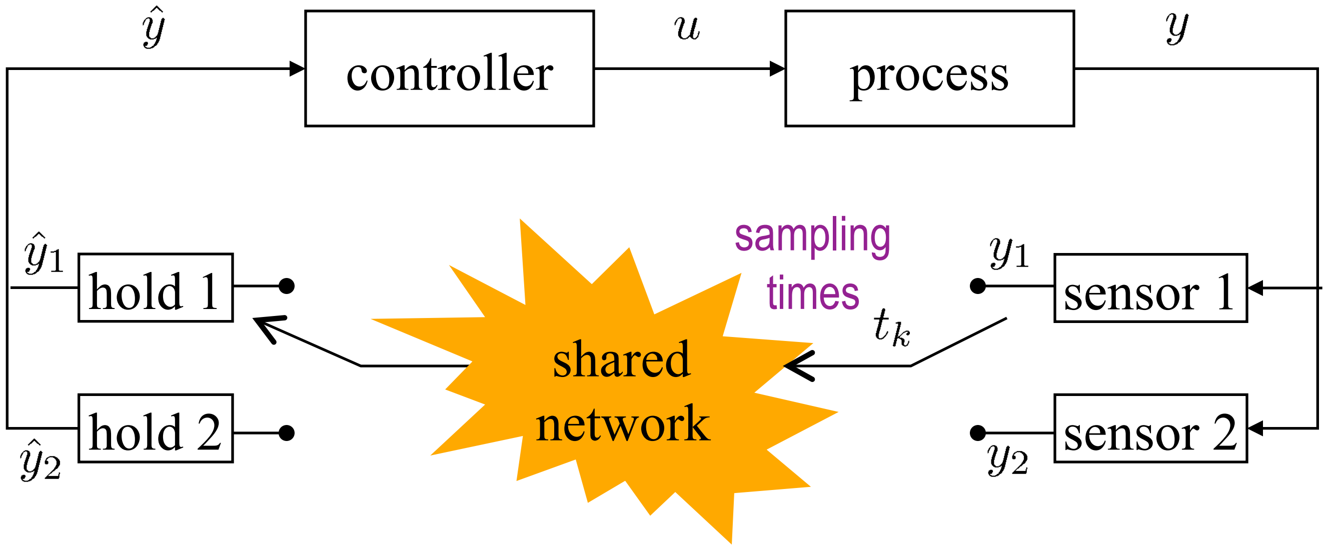
2nd send (old) data from original sampling of continuous-time output

or

2nd send (latest) data from current sampling of continuous-time output

⇒ intersampling times $t_{k+1} - t_k$ typically become random variables

Example I: Networked Control System



$\hat{y}$

controller

u

process

y

$\hat{y}_1$ hold 1

$\hat{y}_2$ hold 2

shared network

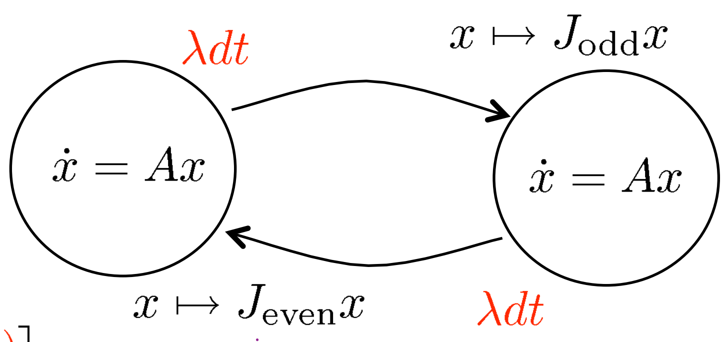
sampling times t_k

y_1 sensor 1

y_2 sensor 2

Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$



λdt $\quad x \mapsto J_{\text{odd}} x$

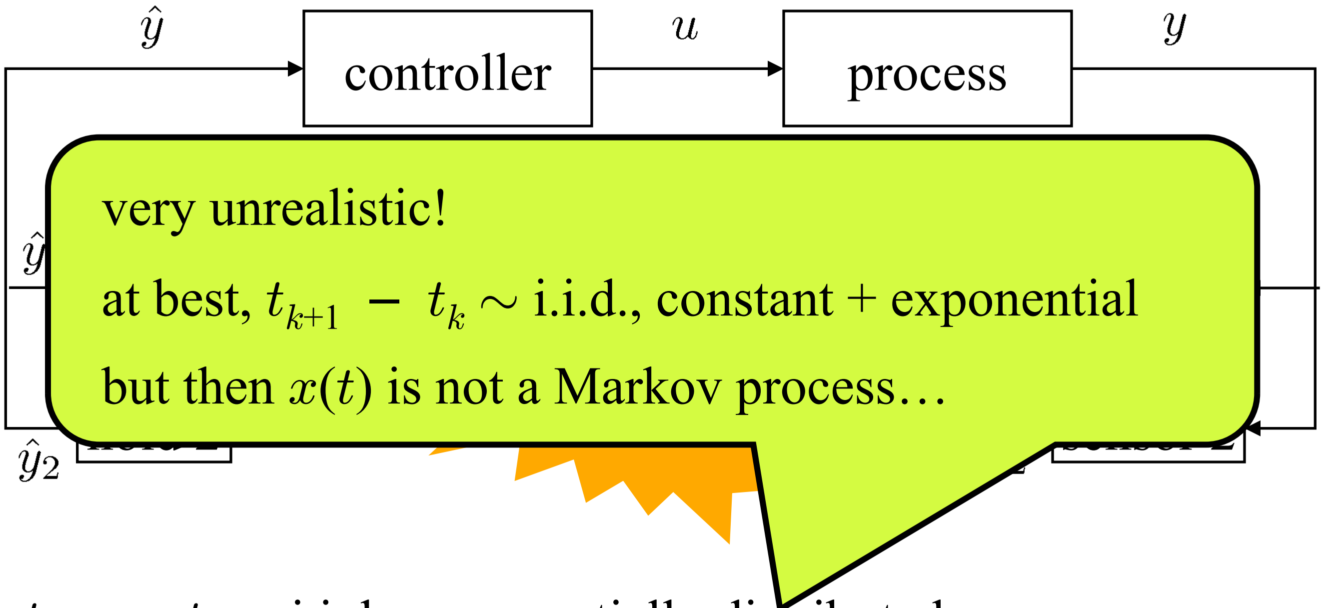
$\dot{x} = Ax$ $\qquad \dot{x} = Ax$

$x \mapsto J_{\text{even}} x$ $\quad \lambda dt$

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$

$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

Example I: Networked Control System

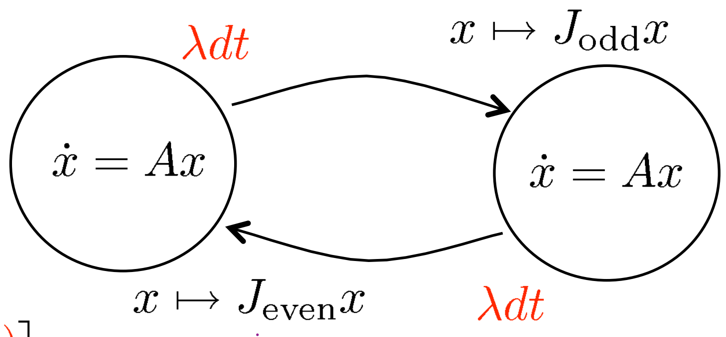


very unrealistic!

at best, $t_{k+1} - t_k \sim$ i.i.d., constant + exponential

but then $x(t)$ is not a Markov process…

Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed



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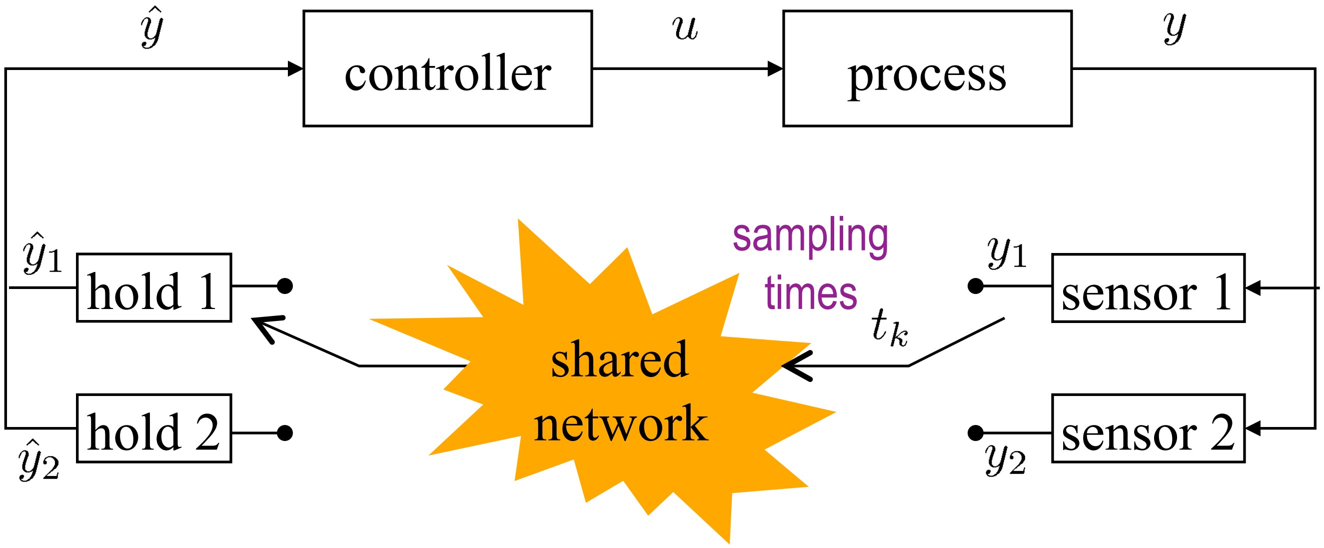
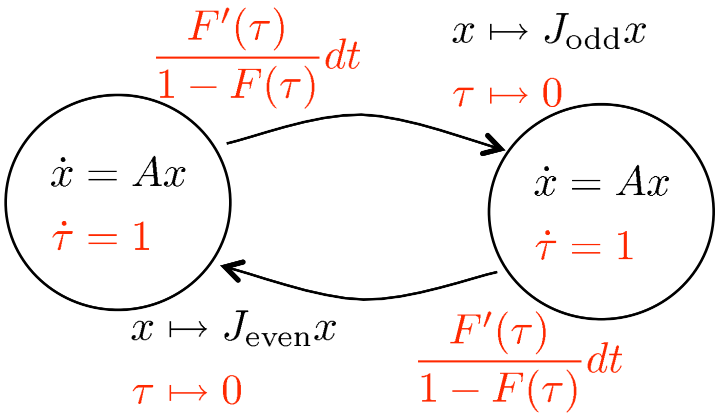
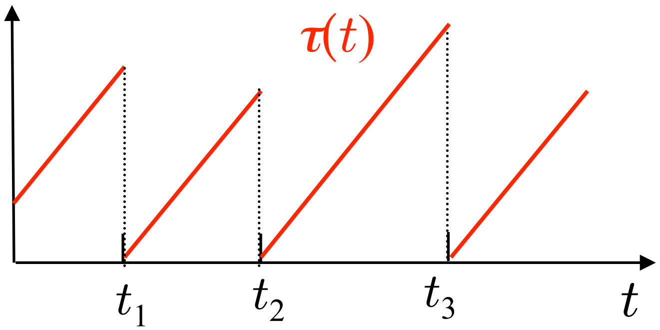
Example I: Networked Control System



Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

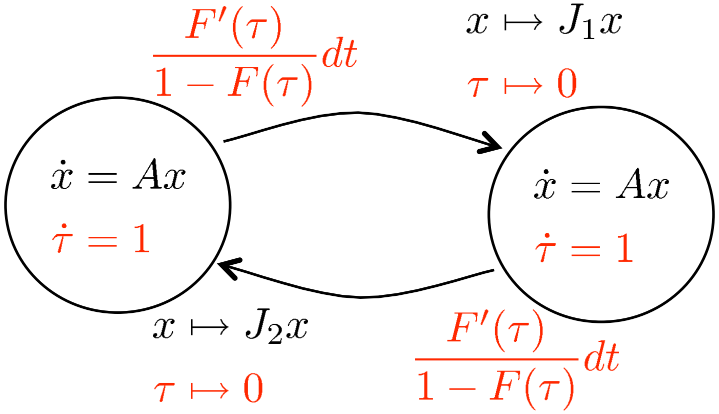




$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

the aggregate state (x,τ) is a Markov process

Impulsive Syst. driven by Renewal Proc.

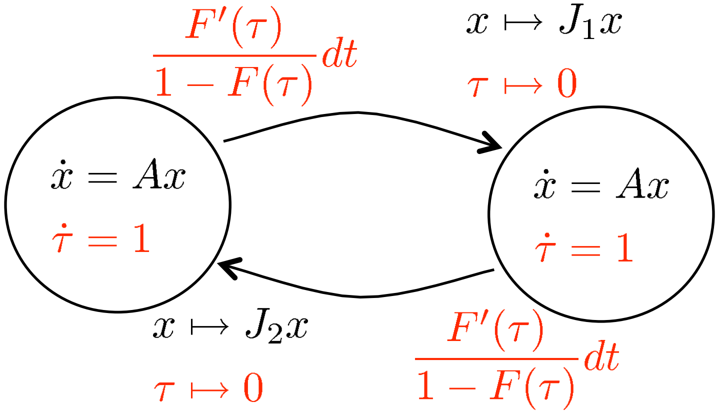


impulsive system ≡ same continuous dynamics for all modes

$N(t)$ ≡ # of jumps before time t

renewal process
(iid inter-increment times)

Impulsive Syst. driven by Renewal Proc.



$\frac{F'(\tau)}{1-F(\tau)}dt$

$x \mapsto J_1 x$

$\tau \mapsto 0$

$\dot{x} = Ax$

$\dot{\tau} = 1$

$\dot{x} = Ax$

$\dot{\tau} = 1$

$x \mapsto J_2 x$

$\tau \mapsto 0$

$\frac{F'(\tau)}{1-F(\tau)}dt$

impulsive system ≡ same continuous
dynamics for all modes

$N(t)$ ≡ # of jumps before time t

renewal process
(iid inter-increment times)

Theorem: (for simplicity, conditions stated for equal reset matrices: $J_1 = J_2 \in \mathbb{R}^{n\times n}$)

system is stochastically stable, i.e., $\displaystyle\int_0^\infty \mathrm{E}[\|x(t)\|^2]dt < \infty$

$$\Updownarrow$$

$$\mathrm{E}_{F(T)}\left[\int_0^T e^{A't}e^{At}dt\right] < \infty$$

and

LMI on $P_{n\times n}$

$$\exists P > 0 : \mathrm{E}_{F(T)}\left[e^{A'T}J'PJe^{AT}\right] - P < 0$$

or

spectral radius condition
on $n^2\times n^2$ matrix

$$\sigma\left(\mathrm{E}_{F(T)}\left[e^{A'T}J' \otimes e^{A'T}J'\right]\right) < 1$$

expected value
w.r.t. inter-jump times

Kronecker product

[ACC'09]

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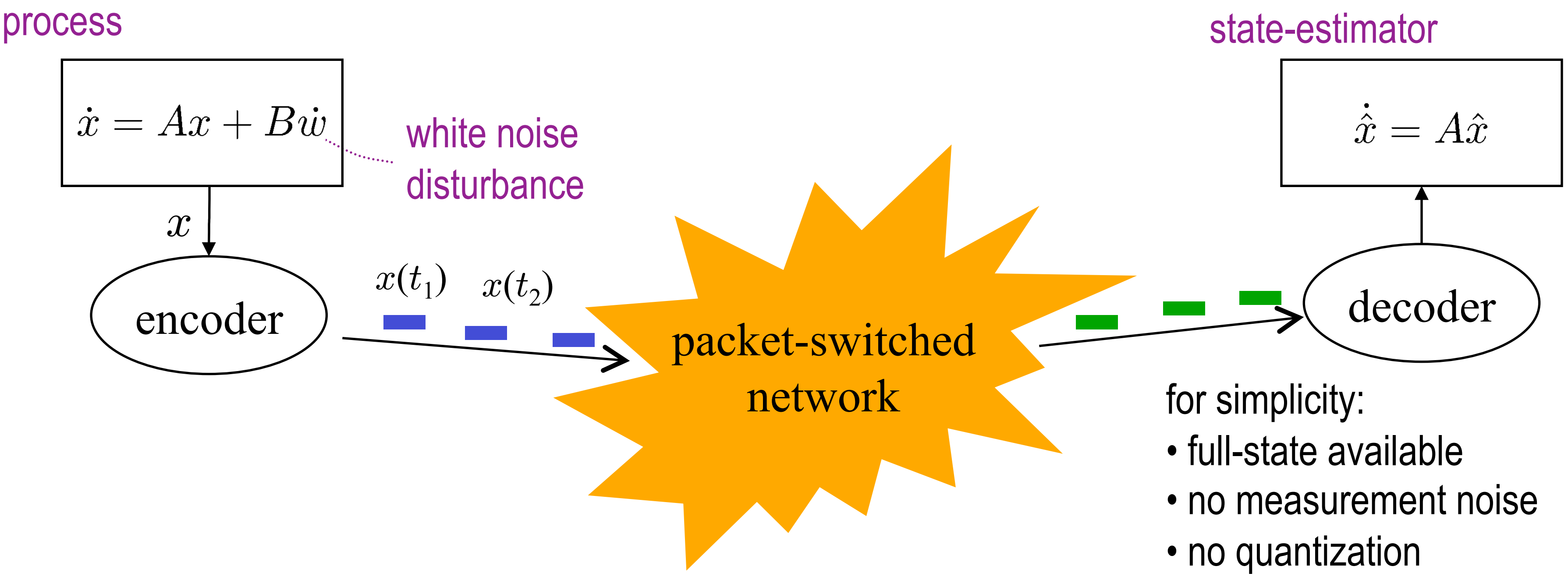
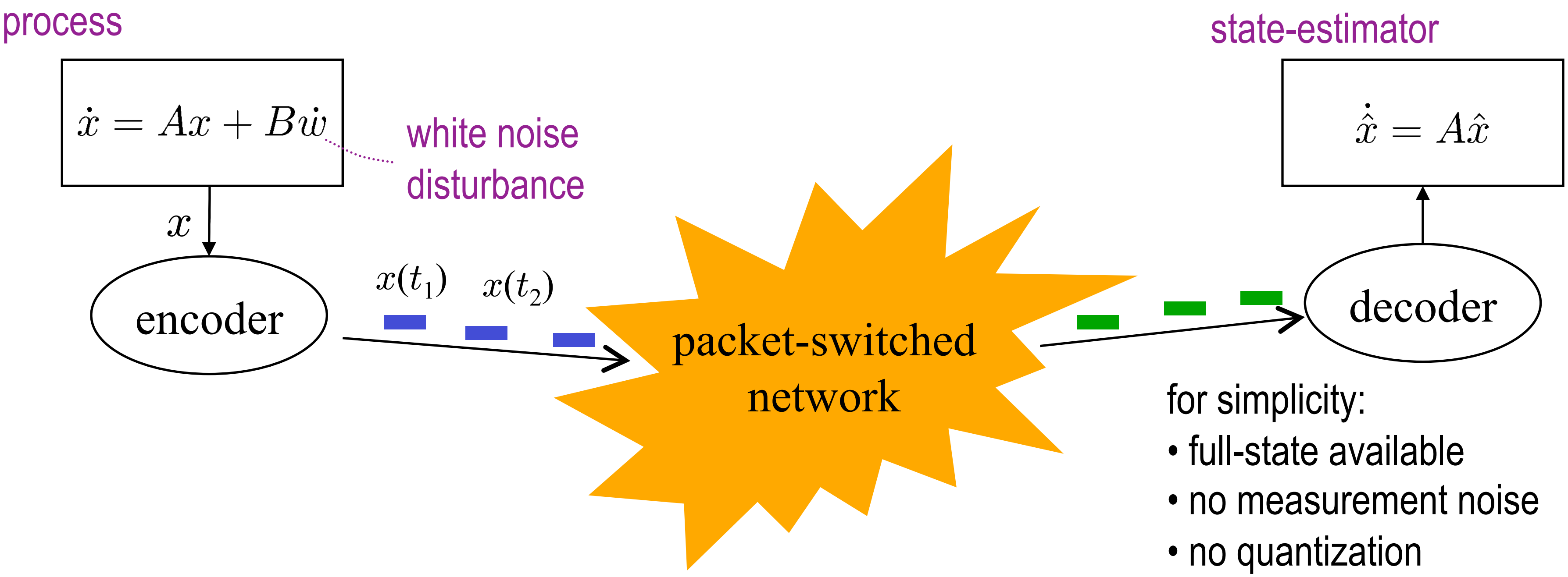
Example II: Estimation through network



for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

encoder logic ≡ determines *when* to send measurements to the network
decoder logic ≡ determines *how* to incorporate received measurements

Stochastic communication logic



for simplicity:
- full-state available
- no measurement noise
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- no transmission delays

encoder logic ≡ determines *when* to send measurements to the network

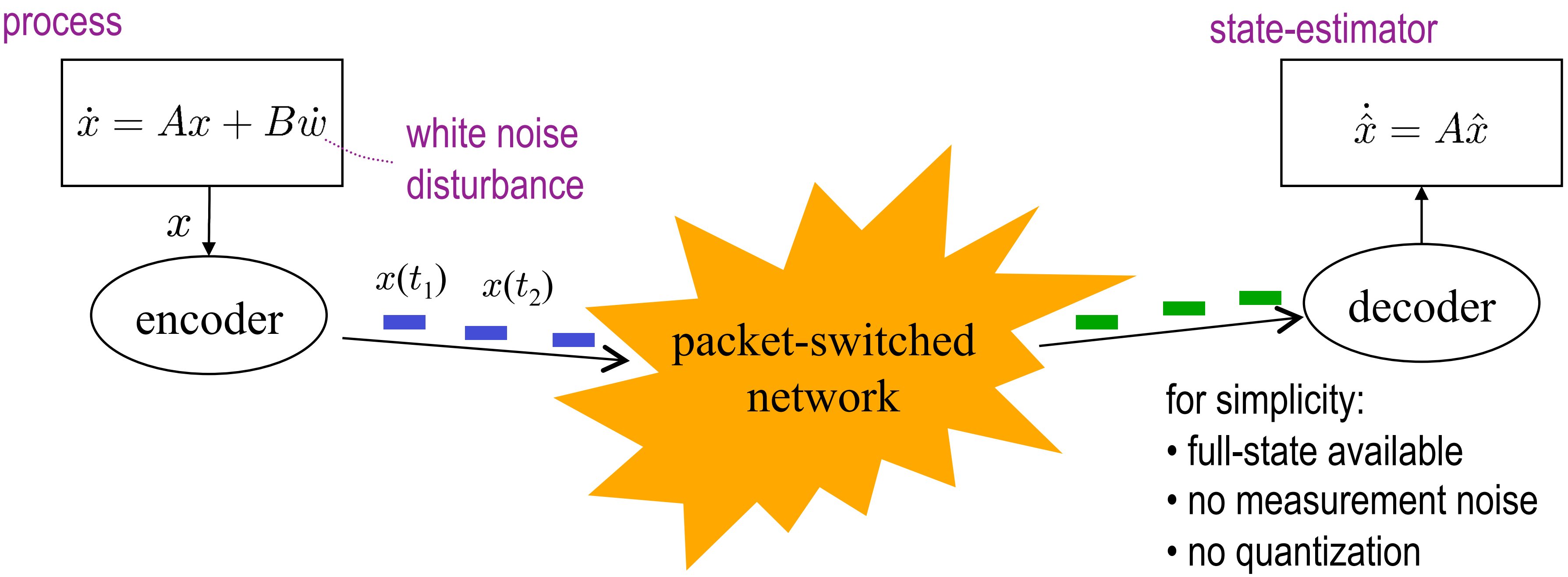
1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from x

decoder logic ≡ determines *how* to incorporate received measurements

4. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

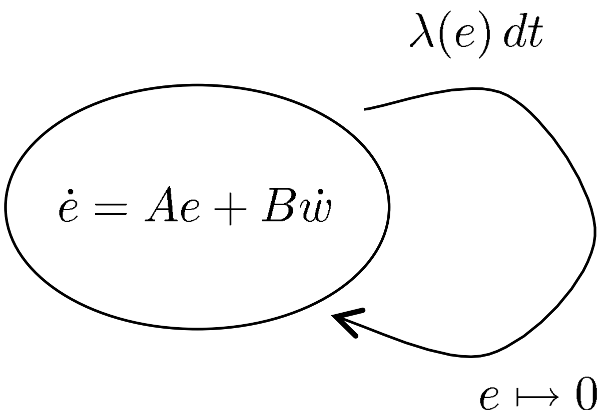
[similar ideas pursued by Astrom, Tilbury, Hristu, Kumar, Basar]

Example II: Remote estimation



for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

Error dynamics: $e := x - \hat{x}$



prob. of sending data in $[t,t+dt)$ depends on current error e

reset error to zero

[CDC’04, CRC Press’06]

Analysis — Lie derivative

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $\psi : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t),t) = \underbrace{\frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t}}_{L_f \psi}$$

derivative
along solution
to ODE

Lie derivative of ψ

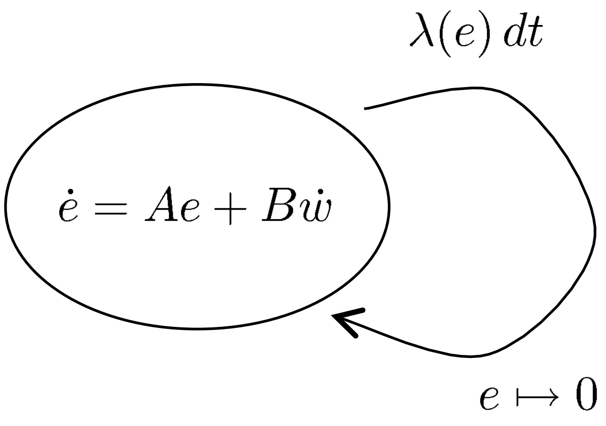
Basis of “Lyapunov” formal arguments to establish boundedness and stability…

E.g., picking $\psi(x,t) := \|x\|^2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t),t) = \frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t} \leq 0 \quad \Rightarrow \quad \psi(x(t),t) = \|x(t)\|^2 \leq \|x(0)\|^2$$

$\|x(t)\|$ remains bounded along trajectories !

Generator of a SHS



prob. of sending data in $[t,t+dt)$
depends on current error e

reset error to zero

Given scalar-valued function $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}[\psi(q,x,t)] = \mathrm{E}\Big[(L\psi)(q,x,t)\Big]}$$

x & q are discontinuous,
but the expected value is
differentiable!!!

Dynkin's formula
(in differential form)

where

$$(L\psi)(e,t) = \frac{\partial\psi}{\partial e}Ae + \frac{\partial\psi}{\partial t}$$
Lie derivative

instantaneous variation

$$+ \Big[\psi(0,t) - \psi(e,t)\Big]\lambda(e)$$
intensity

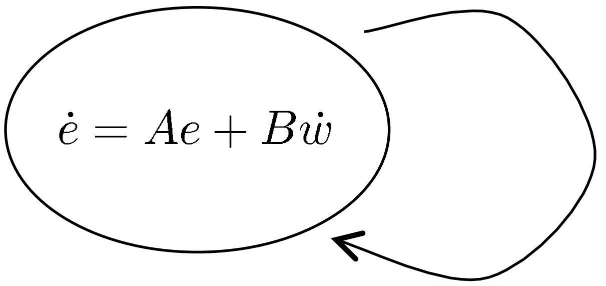
reset term

$$+ \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2\psi}{\partial e^2}B\right)$$
diffusion term

generator
for the SHS

Disclaimer: see *Nonlinear Analysis*'05 for technical assumptions

Generator of a SHS



prob. of sending data in $[t, t+dt)$
depends on current error e

reset e… to zero

generalizes to large classes of
Stochastic Hybrid Systems

Given scalar-valued fu… ψ :

x & q are discontinuous,
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…'s formula
(… differential form)

where

$(L\psi)(e,t) =$ … ∂e Ae … t ………… Lie derivative

instanta…ous variation

generator
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$$+\left[\psi(0,t) - \psi(e,t)\right]\lambda(e)$$ ………… reset term

intensity

$$+\frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2\psi}{\partial e^2}B\right)$$ ………… diffusion term

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Lyapunov-based stability analysis

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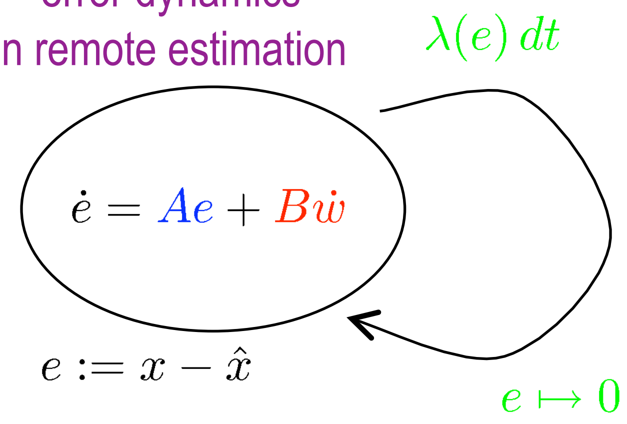
error dynamics
in remote estimation



$$\lambda(e)\,dt$$

$$\dot{e} = Ae + B\dot{w}$$

$$e := x - \hat{x}$$

$$e \mapsto 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}[\psi(e)] = \mathrm{E}\Big[(L\psi)(e)\Big]$$

Dynkin’s formula

$$(L\psi)(e) = \frac{\partial\psi}{\partial e}Ae + \Big[\psi(0) - \psi(e)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2\psi}{\partial e^2}B\right)$$

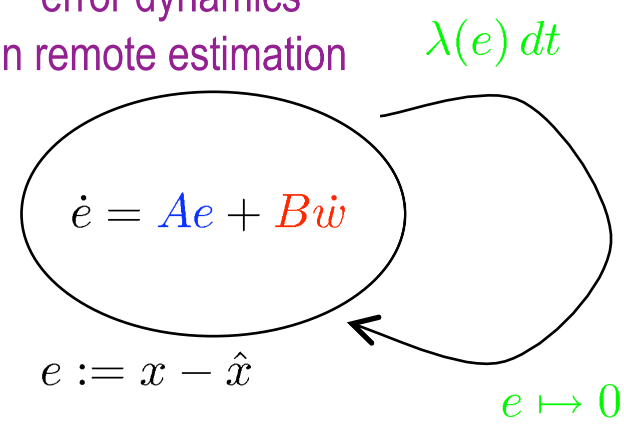
For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $\mathrm{E}[\,e\,] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\,\|\,e\,\|^m\,]$ bounded if and only if $\gamma > m\,\Re[\lambda(A)]$

getting more moments bounded
requires higher comm. rates

Lyapunov-based stability analysis

error dynamics
in remote estimation



$\lambda(e)\,dt$

$\dot{e} = Ae + B\dot{w}$

$e := x - \hat{x}$

$e \mapsto 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\psi(e)] = \mathrm{E}\Big[(L\psi)(e)\Big]$$

Dynkin's formula

$$(L\psi)(e) = \frac{\partial\psi}{\partial e}Ae + \Big[\psi(0) - \psi(e)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\Big(B'\frac{\partial^2\psi}{\partial e^2}B\Big)$$

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $\mathrm{E}[\,e\,] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\,\|\,e\,\|^m\,]$ bounded if and only if $\gamma > m\,\Re[\lambda(A)]$

getting more moments bounded
requires higher comm. rates

For polynomial rates: $\lambda(e) = (e'\,Q\,e)^k$ $\quad Q > 0,\ k > 0$ (reactive transmissions)

1. $\mathrm{E}[\,e\,] \to 0$ (always)
2. $\mathrm{E}[\,\|\,e\,\|^m\,]$ bounded $\forall\, m$

Moreover, one can achieve the same $\mathrm{E}[\,\|e\|^2\,]$
with less communication than with a constant
rate or periodic transmissions…

[CDC'04, Birkhauser'06]

Conclusions

1. A simple SHS model that finds use in several areas (networked control systems, network traffic modeling, biochemistry)
2. The analysis of SHSs is challenging but there are tools available (generator, Lyapunov methods, moment dynamics, truncations)
3. Lots of work to be done:
   - theory
     - ✓ stability/robustness/performance of SHS
   - networked control systems
     - ✓ protocol design to optimize performance & minimize communication resources