WEB APPENDIX A

SOLUTION OF LINEAR EQUATIONS

The purpose of this appendix is to review the methods of solving systems of linear algebraic equations that occur commonly in circuit analysis. Substitution is generally the preferred method for solving two equations with two unknowns, while Cramer's rule offers an easy method for solving three-by-three equations with literal or numeric coefficients by hand. Handheld calculators can solve 2×2, 3×3, or even higher-order equations if the coefficients are numeric. Systems of equations greater than 3×3, especially with literal coefficients, are generally best solved using software tools such as MATLAB. We will discuss how to solve 3×3 equations using Cramer's rule and we will review the matrix algebra approaches for solving more complex systems with software. Actual examples are contained throughout appropriate sections of the text. Appendix B will direct you to specific examples. Circuit analysis often requires solving linear algebraic equations of the type

$$\begin{aligned} 5x_1 - 2x_2 - 3x_3 &= 4 \\ -5x_1 + 7x_2 - 2x_3 &= -10 \\ -3x_1 - 3x_2 + 8x_3 &= 6 \end{aligned} \tag{A–1}$$

where x_1, x_2, and x_3 are unknown voltages or currents. Often some of the unknowns may be missing from one or more of the equations. For example, the equations

$$\begin{aligned} 5x_1 - 2x_2 &= 5 \\ -4x_1 + 7x_2 &= 0 \\ -3x_2 + 8x_3 &= 0 \end{aligned}$$

involve three unknowns with one variable missing in each equation. Such equations can always be put in the standard square form by inserting the missing unknowns with a coefficient of zero.

$$\begin{aligned} 5x_1 - 2x_2 - 0x_3 &= 5 \\ -4x_1 + 7x_2 - 0x_3 &= 0 \\ 0x_1 - 3x_2 + 8x_3 &= 0 \end{aligned} \tag{A–2}$$

Equations (A–1) and (A–2) will be used to illustrate the different methods of solving linear equations.

CRAMER'S RULE

Cramer's rule states that the solution of a system of linear equations for any unknown x_k is found as the ratio of two determinants

$$x_k = \frac{\Delta_k}{\Delta} \qquad \text{(A–3)}$$

where Δ and Δ_k are determinants derived from the given set of equations. A **determinant** is a square array of numbers or symbols called **elements**. The elements are arranged in horizontal rows and vertical columns and are bordered by two vertical straight lines. In general, a determinant contains n^2 elements arranged in n rows and n columns. The value of the determinant is a function of the value and position of its n^2 elements.

The **system determinant** Δ in Eq. (A–3) is made up of the coefficients of the unknowns in the given system of equations. For example, the system determinant for Eq. (A–1) is

$$\Delta = \begin{vmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{vmatrix}$$

and for Eq. (A–2) is

$$\Delta = \begin{vmatrix} 5 & -2 & 0 \\ -4 & 7 & 0 \\ 0 & -3 & 8 \end{vmatrix}$$

These two equations are examples of the general 3×3 determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(A–4)}$$

where a_{ij} is the element in the ith row and jth column.

The determinant Δ_k in Eq. (A–3) is derived from the system determinant by replacing the kth column by the numbers on the right side of the system of equations. For example, Δ_1 for Eq. (A–1) is

$$\Delta_1 = \begin{vmatrix} 4 & -2 & -3 \\ -10 & 7 & -2 \\ 6 & -3 & 8 \end{vmatrix}$$

and Δ_3 for Eq. (A–2) is

$$\Delta_3 = \begin{vmatrix} 5 & -2 & 5 \\ -4 & 7 & 0 \\ 0 & -3 & 0 \end{vmatrix}$$

These examples are 3×3 determinants because the system determinants from which they are derived are 3×3.

In summary, using Cramer's rule to solve linear equations boils down to evaluating the determinants formed using the coefficients of the unknowns and the right side of the system of equations.

EVALUATING DETERMINANTS

The **diagonal rule** gives the value of a 2×2 determinant as the difference in the product of the elements on the main diagonal $(a_{11}a_{22})$ and the product of the elements on the off diagonal $(a_{21}a_{12})$. That is, for a 2×2 determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad \text{(A–5)}$$

The value of 3×3 and higher-order determinants can be found using the method of minors. Every element a_{ij} has a **minor** M_{ij}, which is formed by deleting the row and column containing a_{ij}. For example, the minor M_{21} of the general 3×3 determinant in Eq. (A–4) is

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13}$$

The **cofactor** C_{ij} of the element a_{ij} is its minor M_{ij} multiplied by $(-1)^{i+j}$.

$$C_{ij} = (-1)^{i+j}M_{ij}$$

The signs of the cofactors alternate along any row or column. The appropriate sign for cofactor C_{ij} is found by starting in position a_{11} and counting plus, minus, plus, minus . . . along any combination of rows or columns leading to the position a_{ij}.

To use the **method of minors** we select one (and only one) row or column. The determinant is the sum of the products of the elements in the selected row or column and their cofactors. For example, selecting the first column in Eq. (A–4), we obtain Δ as follows:

$$\begin{aligned}\Delta &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{11}(-1)^2\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^3\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^4\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})\end{aligned}$$

An identical expression for Δ is obtained using any other row or column. For determinants greater than 3×3 the minors themselves can be evaluated using this approach. However, a system of equations leading to determinants larger than 3×3 is probably better handled using computer tools.

EXAMPLE A–1

Solve for the three unknowns in Eq. (A–1) using Cramer's rule.

SOLUTION:

Expanding the system determinant about the first column yields

$$\begin{aligned}\Delta &= \begin{vmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{vmatrix} = 5\begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-5)\begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + (-3)\begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix} \\ &= 5[7 \times 8 - (-2)(-3)] - (-5)[(-2) \times 8 - (-3)(-3)] \\ &\quad + (-3)[(-2)(-2) - (7)(-3)] \\ &= 250 - 125 - 75 = 50\end{aligned}$$

Expanding Δ_1 about the first column yields

$$\Delta_1 = \begin{vmatrix} 4 & -2 & -3 \\ -10 & 7 & -2 \\ 6 & -3 & 8 \end{vmatrix} = 4\begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-10)\begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + (6)\begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix}$$

$$= 200 - 250 + 150 = 100$$

Expanding Δ_2 about the first column yields

$$\Delta_2 = \begin{vmatrix} 5 & 4 & -3 \\ -5 & -10 & -2 \\ -3 & 6 & 8 \end{vmatrix} = 5\begin{vmatrix} -10 & -2 \\ 6 & 8 \end{vmatrix} - (-5)\begin{vmatrix} 4 & -3 \\ 6 & 8 \end{vmatrix} + (-3)\begin{vmatrix} 4 & -3 \\ -10 & -2 \end{vmatrix}$$

$$= -340 + 250 + 114 = 24$$

Expanding Δ_3 about the first column yields

$$\Delta_3 = \begin{vmatrix} 5 & -2 & 4 \\ -5 & 7 & -10 \\ -3 & -3 & 6 \end{vmatrix} = 5\begin{vmatrix} 7 & -10 \\ -3 & 6 \end{vmatrix} - (-5)\begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} + (-3)\begin{vmatrix} -2 & 4 \\ 7 & -10 \end{vmatrix}$$

$$= 60 - 0 + 24 = 84$$

Now, applying Cramer's rule, we solve for the three unknowns.

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{100}{50} = 2$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{24}{50} = 0.48$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{84}{50} = 1.68$$

■

Exercise A–1

Evaluate $\Delta, \Delta_1, \Delta_2$, and Δ_3 for Eq. (A–2).

Answer: 216; 280; 160; 60

MATRICES AND LINEAR EQUATIONS

Circuit equations can be formulated and solved in matrix format. By definition, a **matrix** is a rectangular array written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \qquad \text{(A–6)}$$

The matrix $\mathbf{A}$ in Eq. (A–6) contains m rows and n columns and is said to be of order m *by* n (or $m \times n$). The matrix notation in Eq. (A–6) can be abbreviated as follows:

$$\mathbf{A} = [a_{ij}]_{mn} \qquad \text{(A–7)}$$

where a_{ij} is the element in the ith row and jth column.

SOME DEFINITIONS

Different types of matrices have special names. A **row matrix** has only one row ($m = 1$) and any number of columns. A **column matrix** has only one column ($n = 1$) and any number of rows. A **square matrix** has the same number of rows as columns ($m = n$). A **diagonal matrix** is a square matrix in which all elements not on the main diagonal are zero ($a_{ij} = 0$ for $i \neq j$). An **identity matrix** is a diagonal matrix for which the main diagonal elements are all unity ($a_{ii} = 1$).

For example, given

$$\mathbf{A} = [1 \quad -2 \quad 0 \quad 4] \quad \mathbf{B} = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -7 \\ -3 & 12 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we say that $\mathbf{A}$ is a 1×4 row matrix, $\mathbf{B}$ is a 4×1 column matrix, $\mathbf{C}$ is a 3×3 square matrix, and $\mathbf{U}$ is a 4×4 identity matrix.

The **determinant** of a square matrix $\mathbf{A}$ (denoted det $\mathbf{A}$) has the same elements as the matrix itself. For example, given

$$\mathbf{A} = \begin{bmatrix} 4 & -6 \\ 1 & -2 \end{bmatrix} \text{ then } \det \mathbf{A} = \begin{vmatrix} 4 & -6 \\ 1 & -2 \end{vmatrix} = -8 + 6 = -2$$

The **transpose** of a matrix $\mathbf{A}$ (denoted $\mathbf{A}^T$) is formed by interchanging the rows and columns. For example, given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 8 \\ 4 & 7 & -1 & -3 \end{bmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 0 & -1 \\ 8 & -3 \end{bmatrix}$$

The **adjoint** of a square matrix $\mathbf{A}$ (denoted adj $\mathbf{A}$) is formed by replacing each element a_{ij} by its cofactor C_{ij} and then transposing.

$$\text{adj } \mathbf{A} = [C_{ij}]^T \tag{A–8}$$

For example, if

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 0 & 5 \end{bmatrix} \text{ then } C_{11} = 5 \; C_{12} = 0 \; C_{21} = -2 \; C_{22} = -3$$

and therefore

$$\text{adj } \mathbf{A} = \begin{bmatrix} 5 & 0 \\ -2 & -3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ 0 & -3 \end{bmatrix}$$

MATRIX ALGEBRA

The matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if and only if they have the same number of rows and columns, and $a_{ij} = b_{ij}$ for all i and j. Matrix addition is possible only when two matrices have the same number of rows and columns. When two matrices are of the same order, their sum is obtained by adding the corresponding elements; that is,

$$\text{If } \mathbf{C} = \mathbf{A} + \mathbf{B} \text{ then } c_{ij} = a_{ij} + b_{ij} \tag{A–9}$$

For example, given

$$\mathbf{A} = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 0 \\ 2 & -4 \end{bmatrix} \text{ then } \mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 4 \\ -1 & -6 \end{bmatrix}$$

Multiplying a matrix $\mathbf{A}$ by a scalar constant k is accomplished by multiplying every element by k; that is, $k\mathbf{A} = [ka_{ij}]$. In particular, if $k = -1$ then $-\mathbf{B} = [-b_{ij}]$, and applying the matrix addition rule yields matrix **subtraction**.

$$\text{If } \mathbf{C} = \mathbf{A} - \mathbf{B} \text{ then } c_{ij} = a_{ij} - b_{ij} \tag{A–10}$$

Multiplication of two matrices $\mathbf{AB}$ is defined only if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$. In general, if $\mathbf{A}$ is of order $m \times n$ and $\mathbf{B}$ is of order $n \times r$, then the product $\mathbf{C} = \mathbf{AB}$ is a matrix of order $m \times r$. The element c_{ij} is found by summing the products of the elements in the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$.

$$c_{ij} = [a_{i1}\, a_{i2}\, \ldots\, a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \tag{A–11}$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

In other words, matrix multiplication is a row-by-column operation.

Matrix multiplication is not commutative, so usually $\mathbf{AB} \neq \mathbf{BA}$. Two important exceptions are (1) the product of a square matrix $\mathbf{A}$ and an identity matrix $\mathbf{U}$ for which $\mathbf{UA} = \mathbf{AU} = \mathbf{A}$, and (2) the product of a square matrix $\mathbf{A}$ and its **inverse** (denoted $\mathbf{A}^{-1}$) for which $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{U}$. A closed-form formula for the inverse of a square matrix is

$$\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{\det \mathbf{A}} \tag{A–12}$$

That is, the inverse can be found by multiplying the adjoint matrix of $\mathbf{A}$ by the scalar $1/\det \mathbf{A}$. If $\det \mathbf{A} = 0$, then $\mathbf{A}$ is said to be **singular** and $\mathbf{A}^{-1}$ does not exist. Equation (A–12) is useful for deriving properties of the inverse of a matrix. It is not, however, a very efficient way to calculate the inverse of a matrix of order greater than 3×3.

Exercise A–2

Given:

$$\mathbf{A} = \begin{bmatrix} -5 & 7 \\ 7 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$$

Calculate $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^{-1}$, and $\mathbf{B}^{-1}$.

Answers:

$$\mathbf{AB} = \begin{bmatrix} 27 & -9 \\ 87 & -29 \end{bmatrix} \quad \mathbf{BA} = \begin{bmatrix} -22 & 10 \\ -44 & 20 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{104}\begin{bmatrix} -11 & 7 \\ 7 & 5 \end{bmatrix} \quad \mathbf{B}^{-1} \text{ does not exist}$$

MATRIX SOLUTION OF LINEAR EQUATIONS

The three linear equations in Eq. (A–1) are

$$\begin{aligned} 5x_1 - 2x_2 - 3x_3 &= 4 \\ -5x_1 + 7x_2 - 2x_3 &= -10 \\ -3x_1 - 3x_2 + 8x_3 &= 6 \end{aligned}$$

These equations are expressed in matrix form as follows:

$$\begin{bmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 6 \end{bmatrix} \tag{A–13}$$

The left side of Eq. (A–13) is the product of a 3×3 square matrix and a 3×1 column matrix of unknowns. The elements in the square matrix are the coefficients of the unknown in the given equations. The matrix product on the left side in Eq. (A–13) produces a 3×1 matrix, which equals the 3×1 column matrix on the right side. The elements of the 3×1 on the right side are the constants on the right sides of the given equations.

In symbolic form we write the matrix equation in Eq. (A–13) as

$$\mathbf{AX} = \mathbf{B} \tag{A–14}$$

where

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 4 \\ -10 \\ 6 \end{bmatrix}$$

Left multiplying Eq. (A–14) by $\mathbf{A}^{-1}$ yields

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

But by definition $\mathbf{A}^{-1}\mathbf{A} = \mathbf{U}$ and $\mathbf{UX} = \mathbf{X}$; therefore

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \tag{A–15}$$

To solve linear equations by matrix methods, we calculate the product $\mathbf{A}^{-1}\mathbf{B}$.

To implement the matrix approach, we must first find $\mathbf{A}^{-1}$ using Eq. (A–12). The determinant of the coefficient matrix is

$$\det \mathbf{A} = \begin{bmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{bmatrix} = 50$$

The cofactors of the first row of the coefficient matrix are

$$C_{11} = -\begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} = 50 \quad C_{12} = \begin{vmatrix} -5 & -2 \\ -3 & 8 \end{vmatrix} = 46$$

$$C_{13} = -\begin{vmatrix} -5 & 7 \\ -3 & -3 \end{vmatrix} = 36$$

The cofactors for the second and third rows are

$$\begin{matrix} C_{21} = 25 & C_{22} = 31 & C_{23} = 21 \\ C_{31} = 25 & C_{32} = 25 & C_{33} = 25 \end{matrix}$$

Now, using Eq. (A–12), we obtain $\mathbf{A}^{-1}$ as

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}} = \frac{1}{50}\begin{bmatrix} 50 & 46 & 36 \\ 25 & 31 & 21 \\ 25 & 25 & 25 \end{bmatrix}^T = \frac{1}{50}\begin{bmatrix} 50 & 25 & 25 \\ 46 & 31 & 25 \\ 36 & 21 & 25 \end{bmatrix}$$

Using Eq. (A–15), we solve for the column matrix of unknowns as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{50}\begin{bmatrix} 50 & 25 & 25 \\ 46 & 31 & 25 \\ 36 & 21 & 25 \end{bmatrix}\begin{bmatrix} 4 \\ -10 \\ 6 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 100 \\ 24 \\ 84 \end{bmatrix}$$

which yields $x_1 = 2$, $x_2 = 24/50$, and $x_3 = 84/50$. These are, of course, the same results previously obtained using Cramer's rule.