Index Notation for Vector Calculus

by

Ilan Ben-Yaacov and Francesc Roig

Index notation, also commonly known as subscript notation or tensor notation, is an extremely useful tool for performing vector algebra. Consider the coordinate system illustrated in Figure 1. Instead of using the typical axis labels x, y, and z, we use x_1, x_2, and x_3, or

$$x_i \qquad i = 1, 2, 3$$

The corresponding unit basis vectors are then $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, or

$$\hat{e}_i \qquad i = 1, 2, 3$$

The basis vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ have the following properties:

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1 \tag{1}$$

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0 \tag{2}$$

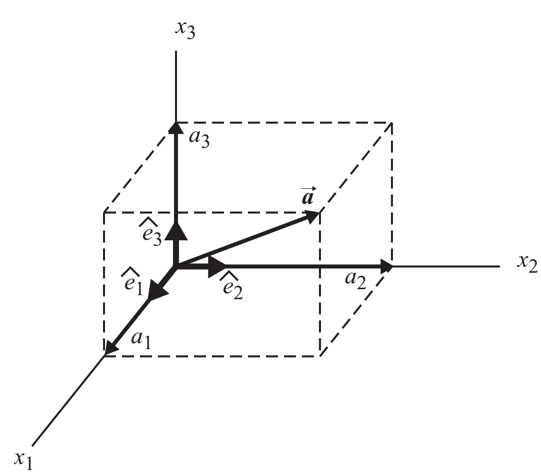

Figure 1: Reference coordinate system.

We now introduce the **Kronecker delta** symbol δ_{ij}. δ_{ij} has the following properties:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i,j = 1,2,3 \tag{3}$$

Using Eqn 3, Eqns 1 and 2 may be written in index notation as follows:

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}} \qquad i,j = 1,2,3 \tag{4}$$

In standard vector notation, a vector $\vec{\mathrm{A}}$ may be written in component form as

$$\vec{\mathrm{A}} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \tag{5}$$

Using index notation, we can express the vector $\vec{\mathrm{A}}$ as

$$\begin{aligned} \vec{\mathrm{A}} &= A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \\ &= \sum_{i=1}^{3} A_i \hat{e}_i \end{aligned} \tag{6}$$

Notice that in the expression within the summation, the index i is *repeated*. Repeated indices are always contained within summations, or phrased differently a repeated index *implies* a summation. Therefore, the summation symbol is typically dropped, so that $\vec{\mathrm{A}}$ can be expressed as

$$\vec{\mathrm{A}} = A_i \hat{e}_i \equiv \sum_{i=1}^{3} A_i \hat{e}_i \tag{7}$$

This repeated index notation is known as Einstein's convention. Any repeated index is called a *dummy index*. Since a repeated index implies a summation over all possible values of the index, one can always relabel a dummy index, *i.e.*

$$\begin{aligned} \vec{\mathrm{A}} &= A_i \hat{e}_i = A_j \hat{e}_j = A_k \hat{e}_k \quad etc. \\ &\equiv A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \end{aligned} \tag{8}$$

The Scalar Product in Index Notation

We now show how to express *scalar products* (also known as inner products or dot products) using index notation. Consider the vectors $\vec{a}$ and $\vec{b}$, which can be expressed using index notation as

$$\begin{aligned} \vec{a} &= a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 = a_i\hat{e}_i \\ \vec{b} &= b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3 = b_j\hat{e}_j \end{aligned} \tag{9}$$

Note that we use different indices (i and j) for the two vectors to indicate that the index for $\vec{b}$ is completely independent of that used for $\vec{a}$. We will first write out the scalar product $\vec{a} \cdot \vec{b}$ in long-hand form, and then express it more compactly using some of the properties of index notation.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \left(\sum_{i=1}^{3} a_i\hat{e}_i\right) \cdot \left(\sum_{j=1}^{3} b_j\hat{e}_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left[(a_i\hat{e}_i) \cdot (b_j\hat{e}_j)\right] \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left[a_ib_j\left(\hat{e}_i \cdot \hat{e}_j\right)\right] \qquad \text{(commutative property)} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left(a_ib_j\delta_{ij}\right) \qquad \text{(from Eqn 3)} \end{aligned}$$

Summing over all values of i and j, we get

$$\begin{aligned} \vec{a} \cdot \vec{b} &= a_1b_1\delta_{11} + a_1b_2\delta_{12} + a_1b_3\delta_{13} \\ &\quad + a_2b_1\delta_{21} + a_2b_2\delta_{22} + a_2b_3\delta_{23} \\ &\quad + a_3b_1\delta_{31} + a_3b_2\delta_{32} + a_3b_3\delta_{33} \\ &= a_1b_1\delta_{11} + a_2b_2\delta_{22} + a_3b_3\delta_{33} \\ &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= \sum_{i=1}^{3} a_ib_i \\ &= a_ib_i = a_jb_j = a_kb_k \end{aligned}$$

Doing this in a more compact notation gives us

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_i\hat{e}_i)\cdot(b_j\hat{e}_j)\\ &= a_ib_j\hat{e}_i\cdot\hat{e}_j = a_ib_j\delta_{ij}\\ &= a_ib_i = a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

Notice that when we have an expression containing δ_{ij}, we simply get rid of the δ_{ij} and set $i = j$ everywhere in the expression.

Example 1: Kronecker delta reduction

Reduce $\delta_{ij}\delta_{jk}\delta_{ki}$:

$$\begin{aligned}\delta_{ij}\delta_{jk}\delta_{ki} &= \delta_{ik}\delta_{ki} \quad (\text{remove } \delta_{ij},\ \text{set } j = i \text{ everywhere})\\ &= \delta ii \quad (\text{remove } \delta_{ik},\ \text{set } k = i \text{ everywhere})\\ &= \sum_{i=1}^{3}\delta_{ii} = \sum_{i=1}^{3} 1 = 1+1+1 = 3\end{aligned}$$

Here we can see that

$$\boxed{\delta_{ii} = 3} \qquad (\text{Einstein convention implied}) \tag{10}$$

Note also that

$$\boxed{\delta_{ij}\delta_{jk} = \delta_{ik}} \tag{11}$$

Example 2: $\vec{r}$ and $\hat{r}$ in index notation

(a) Express $\vec{r}$ using index notation.

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 = \boxed{x_i\hat{e}_i}$$

(b) Express $\hat{r}$ using index notation.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{(\vec{r}\cdot\vec{r})^{1/2}} = \boxed{\frac{x_i\hat{e}_i}{(x_jx_j)^{1/2}}}$$

(c) Express $\vec{a}\cdot\hat{r}$ using index notation.

$$\vec{a}\cdot\hat{r} = \frac{\vec{a}\cdot\vec{r}}{|\vec{r}|} = \boxed{\frac{a_ix_i}{(x_jx_j)^{1/2}}}$$

The Cross Product in Index Notation

Consider again the coordinate system in Figure 1. Using the conventional right-hand rule for cross products, we have

$$\begin{aligned}
&\hat{e}_1\times\hat{e}_1 = \hat{e}_2\times\hat{e}_2 = \hat{e}_3\times\hat{e}_3 = 0\\
&\hat{e}_1\times\hat{e}_2 = \hat{e}_3 \qquad \hat{e}_2\times\hat{e}_1 = -\,\hat{e}_3\\
&\hat{e}_2\times\hat{e}_3 = \hat{e}_1 \qquad \hat{e}_3\times\hat{e}_2 = -\,\hat{e}_1\\
&\hat{e}_3\times\hat{e}_1 = \hat{e}_2 \qquad \hat{e}_1\times\hat{e}_3 = -\,\hat{e}_2
\end{aligned} \tag{12}$$

To write the expressions in Eqn 12 using index notation, we must introduce the symbol ϵ_{ijk}, which is commonly known as the Levi-Civita tensor, the alternating unit tensor, or the permutation symbol (in this text it will be referred to as the permutation symbol). ϵ_{ijk} has the following properties:

$$\begin{array}{ll}
\epsilon_{ijk} = 1 & \text{if } (ijk) \text{ is an even (cyclic) permutation}\\
 & \text{of (123), } i.e.\ \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1\\
\epsilon_{ijk} = -1 & \text{if } (ijk) \text{ is an odd (noncyclic) permutation}\\
 & \text{of (123), } i.e.\ \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1\\
\epsilon_{ijk} = 0 & \text{if two or more subscripts are the same,}\\
 & i.e.\ \epsilon_{111} = \epsilon_{112} = \epsilon_{313} = 0 \quad etc.
\end{array}$$

Hence, we may rewrite the expressions in Eqn 12 as follows:

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \epsilon_{123}\hat{e}_3 \qquad & \hat{e}_2 \times \hat{e}_1 &= \epsilon_{213}\hat{e}_3 \\ \hat{e}_2 \times \hat{e}_3 &= \epsilon_{231}\hat{e}_1 \qquad & \hat{e}_3 \times \hat{e}_2 &= \epsilon_{321}\hat{e}_1 \\ \hat{e}_3 \times \hat{e}_1 &= \epsilon_{312}\hat{e}_2 \qquad & \hat{e}_1 \times \hat{e}_3 &= \epsilon_{132}\hat{e}_2 \end{aligned} \tag{13}$$

Now, we may write a single generalized expression for all the terms in Eqn 13:

$$\boxed{\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k} \tag{14}$$

Here $\epsilon_{ijk}\hat{e}_k \equiv \sum_{k=1}^{3}\epsilon_{ijk}\hat{e}_k$ (k is a dummy index). That is, this works because

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 \quad &= \epsilon_{12k}\hat{e}_k = \sum_{k=1}^{3}\epsilon_{12k}\hat{e}_k \\ &= \epsilon_{121}\hat{e}_1 + \epsilon_{122}\hat{e}_2 + \epsilon_{123}\hat{e}_3 = \hat{e}_3 \end{aligned}$$

The same is true for all of the other expressions in Eqn 13. Note that $\hat{e}_i \times \hat{e}_i = \epsilon_{iik}\hat{e}_k = 0$, since ϵ_{iik} for all values of k. ϵ_{ijk} is also given by the following formula.

$$\boxed{\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)} \qquad i, j, k = 1, 2, 3 \tag{15}$$

This is a remarkable formula that works for ϵ_{ijk} if you do not want to calculate the parity of the permutation (ijk). Also note the following property of ϵ_{ijk}:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji}$$

$i.e.$ switching any two subscripts reverses the sign of the permutation symbol (or in other words ϵ_{ijk} is *anti-symmetric*). Also,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

$i.e.$ cyclic permutations of the subscripts do not change the sign of ϵ_{ijk}. These properties also follow from the formula in Eqn 15.

Now, let's consider the cross product of two vectors $\vec{a}$ and $\vec{b}$, where

$$\vec{a} = a_i \hat{e}_i$$
$$\vec{b} = b_j \hat{e}_j$$

Then

$$\vec{a} \times \vec{b} = (a_i \hat{e}_i) \times (b_j \hat{e}_j) = a_i b_j \hat{e}_i \times \hat{e}_j = a_i b_j \epsilon_{ijk} \hat{e}_k$$

Thus we write for the cross product:

$$\boxed{\vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \hat{e}_k} \tag{16}$$

All indices in Eqn 16 are dummy indices (and are therefore summed over) since they are repeated. We can always relabel dummy indices, so Eqn 16 may be written equivalently as

$$\vec{a} \times \vec{b} = \epsilon_{pqr} a_p b_q \hat{e}_r$$

Returning to Eqn 16, the kth component of $\vec{a} \times \vec{b}$ is

$$\left(\vec{a} \times \vec{b}\right)_k = \epsilon_{ijk} a_i b_j$$

where now only i and j are dummy indices. Note that the cross product may also be written in determinant form as follows:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{17}$$

The follwoing is a very important identity involoving the product of two permutation symbols.

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{18}$$

The proof of this identity is as follows:

- If any two of the indices i, j, k or l, m, n are the same, then clearly the left-hand side of Eqn 18 must be zero. This condition would also result in two of the rows or two of the columns in the determinant being the same, so therefore the right-hand side must also equal zero.
- If (i, j, k) and (l, m, n) both equal (1,2,3), then both sides of Eqn 18 are equal to one. The left-hand side will be 1×1, and the right-hand side will be the determinant of the identity matrix.
- If any two of the indices i, j, k or l, m, n are interchanged, the corresponding permutation symbol on the left-hand side will change signs, thus reversing the sign of the left-hand side. On the right-hand side, an interchange of two indices results in an interchange of two rows or two columns in the determinant, thus reversing its sign.

Therefore, all possible combinations of indices result in the two sides of Eqn 18 being equal. Now consider the special case of Eqn 18 where $n = k$. In this case, the repeated index k implies a summation over all values of k. The product of the two permutation symbols is now

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{lmk} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & \delta_{kk} \end{vmatrix} \qquad \text{(note } \delta_{kk} = 3\text{)} \\
&= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} + \delta_{im}\delta_{jk}\delta_{kl} \\
&\quad -\delta_{ik}\delta_{jm}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jk}\delta_{km} \qquad (19) \\
&= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} \\
&\quad -\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} \qquad \text{(from Eqn 11)}
\end{aligned}$$

Or finally

$$\boxed{\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \qquad (20)$$

Eqn 20 is an *extremely* useful property in vector algebra and vector calculus applications. It can also be expressed compactly in determinant form as

$$\epsilon_{ijk}\epsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} \tag{21}$$

The cyclic property of the permutation symbol allows us to write also

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

To recap:

$$\begin{array}{ccc} \hat{e}_i \cdot \hat{e}_j = \delta_{ij} & \text{and} & \vec{a} \cdot \vec{b} = a_i b_i \\ \hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k & \text{and} & \vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \hat{e}_k \end{array}$$

These relationships, along with Eqn 20, allow us to prove any vector identity.

Example 3: The scalar triple product

Show that $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_i \hat{e}_i) \cdot (\epsilon_{jkm} b_j c_k \hat{e}_m) \\ &= \epsilon_{jkm} a_i b_j c_k (\hat{e}_i \cdot \hat{e}_m) \\ &= \epsilon_{jkm} a_i b_j c_k \delta_{im} \\ &= \epsilon_{jki} a_i b_j c_k \end{aligned}$$

or

$$\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{ijk} a_i b_j c_k}$$

From our permutation rules, it follows that

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \epsilon_{ijk} a_i b_j c_k \\ &= \epsilon_{kij} c_k a_i b_j = \boxed{\vec{c} \cdot (\vec{a} \times \vec{b})} \\ &= \epsilon_{jki} b_j c_k a_i = \boxed{\vec{b} \cdot (\vec{c} \times \vec{a})} \end{aligned}$$