Optimal Mechanisms for Robust Coordination in Congestion Games

Philip N. Brown and Jason R. Marden

Abstract—Uninfluenced social systems often exhibit suboptimal performance; specially designed taxes can influence agent choices and thereby bring aggregate social behavior closer to optimal. A perfect system characterization may enable a planner to apply simple taxes to incentivize desirable behavior, but system uncertainties may necessitate highly sophisticated taxation methodologies. Using a model of network routing, we study the effect of system uncertainty on a designer’s ability to influence behavior with financial incentives. We show that, in principle, it is possible to design taxes that guarantee that selfish network flows are arbitrarily close to optimal flows, despite the fact that agents’ tax sensitivities and the network topology are unknown to the designer. In general, these taxes may be large; accordingly, for affine-cost parallel-network routing games, we explicitly derive the optimal bounded tolls and the best-possible performance guarantee as a function of a toll upper bound. Finally, we restrict attention to simple fixed tolls and show that they fail to provide strong performance guarantees if the designer lacks accurate information about the network topology or user sensitivities.

Index Terms—Game theory, Incentives, Traffic congestion.

I. INTRODUCTION

It is well known that in systems that are driven by social behavior, lack of coordination and agents’ self-interested behavior can significantly degrade system performance. This poor performance is commonly referred to as the price of anarchy (PoA), defined as the ratio between the worst-case social welfare resulting from selfish behavior and the optimal social welfare [2]. This degradation of performance due to selfish behavior has been the subject of research in areas of network resource allocation [3], distributed control [4], traffic congestion [5], [6], and others. As a result, there is a growing body of research geared at influencing social behavior to improve system performance [7]–[13].

To study the issues surrounding the problem of influencing selfish social behavior, we turn to a simple model of traffic routing: a mass of traffic needs to be routed across a network in a way that minimizes the average network transit time. If a central planner can direct traffic explicitly, it is straightforward to compute the routing profile that minimizes total congestion. However, in real systems, it may not be possible to implement such direct centralized control or prescribe such optimal coordinated behavior: for example, if the network represents a city’s road network, individual drivers make their own routing choices in response to their own personal objectives.

Accordingly, we may model this routing problem as a nonatomic congestion game, where the traffic can be viewed as a collection of infinitely many users, each controlling an infinitesimally small amount of traffic and seeking to minimize its own transit time. We use the popular concept of a Nash flow (defined as a routing profile in which no user can switch to a different path and decrease her transit delay) to characterize the routing profile resulting from such self-interested behavior. It is widely known that Nash flows can exhibit considerably higher congestion than optimal flows. An important result in this setting states that a Nash flow on a network with general latency functions can be arbitrarily worse than an optimal flow [14]. That is, the PoA is unbounded; this is true even on networks consisting of only two links. Recent research has investigated the PoA of transportation networks under various conditions [15]–[18].

A separate research agenda has investigated methods of incentivizing individual network users to choose more efficient routes, thereby aligning Nash flows with optimal flows. This can be viewed as an attempt to incentivize coordination between the users of the network. A natural approach to this is to charge monetary taxes for the use of network links. Existing research has explored methods of designing such optimal taxes given that the tax designer has access to certain information regarding the system. In [19]–[21], it is shown that optimal “fixed” taxes (i.e., taxes are constant functions of traffic flow) can be computed for any routing game, but the computation requires precise characterizations of the network topology, user demands, and user tax sensitivities. In contrast, the authors of [22] and [23] derive optimal taxes known as “marginal-cost taxes,” which require no knowledge of the network topology or user demands, but require that all users share a common known tax sensitivity. Furthermore, the marginal-cost taxation functions must be strictly flow varying. Section III details these results.

In this paper, we ask if it is possible to compute optimal taxes with minimal information about the system and present several new results showcasing the relationship between available tolling methodologies, uncertainty, and achievable performance. We term this goal “robust coordination,” as we desire to incentivize agents to behave as though they are coordinating with one another, but we require that our behavior-influencing

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mechanisms are robust to mischaracterizations of the system. Since PoA is simply a cost metric in worst case over some set of unknown information, it lends itself naturally to quantifying the robustness of taxation mechanisms to unknown information. Thus, our analysis represents a departure both from the typical descriptive PoA research and from the complete-information assumptions of the taxation literature.

Our main contribution is to derive a universal taxation mechanism that guarantees arbitrarily good performance for any routing game while requiring no prior knowledge of the specific network, user demand profile, or distribution of user sensitivities. That is, our derived taxes are robust to gross mischaracterizations of the above quantities. This result holds for networks with general latency functions and any topology, suggesting that surprisingly little information is required in principle.

Our next result explains the effect of reducing the designer’s capabilities while maintaining a high level of uncertainty. To this end, our second contribution is to explore the effect of placing an upper bound on the allowable tolling functions. This may have practical value in settings where very large tolls may be impossible (or politically unpalatable) to implement. For parallel networks with linear-affine latency functions, we derive the optimal tolling functions that minimize worst-case performance degradation for any unknown distribution of user sensitivities and toll upper bound, requiring no prior knowledge of the number of network links. These optimal tolls are simple affine functions of flow. We show that for parallel networks with linear-affine cost functions and simple user demands, the worst-case performance degradation strictly decreases with the toll upper bound. Our results suggest that large tolls can compensate for a poor characterization of user sensitivities. Unfortunately, by imposing an upper bound on allowable taxation functions, optimal behavior can no longer be guaranteed. Thus, this result additionally implies that unbounded tolls are necessary to enforce optimal flows if both the network topology and user sensitivities are unknown.

Our results in Section VI explore a further restriction on the designer’s capabilities, requiring that tolls do not depend on flow (i.e., requiring fixed tolls rather than tolling functions). These results suggest that fixed tolls lack robustness to mischaracterizations of the network topology and user sensitivity. First, if the network topology is unknown, fixed tolls cannot enforce perfectly optimal routing, and we present a simple setting in which network performance can be arbitrarily bad if fixed tolls are not allowed to depend on the network structure. Finally, we show that even if fixed tolls are allowed to depend on the network topology and user demands, they provide relatively poor performance guarantees when the user sensitivities are unknown. Here, by reducing the designer’s capability (by disallowing access to flow-varying taxation functions), we dramatically reduce the achievable performance guarantees in the presence of uncertainty. That is, fixed tolls are significantly less robust than flow-varying tolls. Our negative result here vividly demonstrates the need for a clear understanding of the robustness of incentive mechanisms to model imperfections.

II. MODEL AND PERFORMANCE METRICS

A. Routing Game

Consider a network routing problem in which a unit mass of traffic needs to be routed across a network $(V, E)$, which consists of a vertex set $V$ and edge set $E \subseteq (V \times V)$. We call a source/destination vertex pair $(s^c, t^c) \in (V \times V)$ a commodity, and the set of all such commodities $C$. For each $c \in C$, there is a mass of traffic $r^c > 0$ that needs to be routed from $s^c$ to $t^c$. We write $P^c \subset 2^E$ to denote the set of paths available to traffic in commodity $c$, where each path $p \in P^c$ consists of a set of edges connecting $s^c$ to $t^c$. Let $P = \bigcup P^c$.

We write $f^c \geq 0$ to denote the mass of traffic from commodity $c$ using path $p$, and $f^c_p \triangleq \sum_{e \in E} f^c_e$. A feasible flow $f \in \mathbb{R}^{|P|}$ is an assignment of traffic to various paths such that for each $c$, $\sum_{p \in P^c} f^c_p = r^c$. Without loss of generality, we assume that $\sum_{c \in C} r^c = 1$.

Given a flow $f$, the flow on edge $e$ is given by $f_e = \sum_{p \in P^e} f^c_p$. To characterize transit delay as a function of traffic flow, each edge $e \in E$ is associated with a specific latency function $\ell_e : [0, 1] \to [0, \infty)$. We adopt the standard assumptions that latency functions are nondecreasing, continuously differentiable, and convex. Note that latency functions are anonymous: all traffic affects delay equally. The cost of a flow $f$ is measured by the total latency, given by

$$
\mathcal{L}(f) = \sum_{e \in E} f_e \cdot \ell_e(f_e) = \sum_{p \in P} f^c_p \cdot \ell_p(f)
$$

where $\ell_p(f) = \sum_{e \in P^p} \ell_e(f_e)$ denotes the latency on path $p$. We denote the flow that minimizes the total latency by

$$
f^* \in \arg \min_{f \text{ is feasible}} \mathcal{L}(f).
$$

A routing problem is given by the tuple $G = (V, E, C, \{\ell_e\})$. We write the set of all such routing problems as $\mathcal{G}$ and often write $c \in G$ to denote $(c \in G : G \in \mathcal{G})$.

In this paper, we study taxation mechanisms for influencing the emergent collective behavior resulting from self-interested price-sensitive users. To that end, we model the above routing problem as a nonatomic game in which the traffic models a large population of users. Thus, we use the terms “traffic,” “users,” and “agents” interchangeably. We assign each edge $e \in E$ a flow-dependent nondecreasing taxation function $\tau_e : [0, 1] \to \mathbb{R}^+$. We characterize the taxation sensitivities of the users in commodity $c$ with a monotone nondecreasing function $s^c : [0, r^c] \to [S_L, S_U]$, where each user $x \in [0, r^c]$ has a taxation sensitivity $s^c_x \in [S_L, S_U] \subseteq \mathbb{R}^+$ and $S_L \geq S_L \geq 0$ denote upper and lower sensitivity bounds, respectively. Given a flow $f$, the cost that user $x \in [0, r^c]$ experiences for using path $p \in P^c$ is of the form

$$
J^c_x(f) = \sum_{e \in P^c} [\ell_e(f_e) + s^c_x \tau_e(f_e)].
$$

Thus, for each user $x \in [0, r^c]$, the sensitivity $s^c_x$ can be viewed as a constant gain on the toll; a user’s experienced cost is then the sum of the latency and sensitivity-weighted toll. Note that sensitivity can be interpreted as the reciprocal of an agent’s value of time.\footnote{Note that we allow all edges to be taxed (as in [19]–[23]); see [24] for a relaxation of this requirement.} Note that constant sensitivity is a commonly studied special case; alternative formulations are possible [25].
source–destination paths. We call a flow $f$ a Nash flow if for all commodities $c \in C$ and all users $x \in [0, r^c]$, we have

$$J^c_e (f) = \min_{p \in P} \left\{ \sum_{r \in r^c} [\ell_e (f_r) + s^c_x \tau_e (f_r)] \right\}. \quad (4)$$

It is well known that a Nash flow exists for any nonatomic game of the above form [26].

In our analysis, we assume that each sensitivity distribution function $s^c_x$ is unknown; for a given routing problem $G$ and $S_L^c \geq S_L \geq 0$, we define the set of possible sensitivity distributions as the set of monotone nondecreasing functions $\delta_G = \{ s^c : [0, r^c] \rightarrow [S_L^c, S_L] \}_{c \in C}$. We write $s \in \delta_G$ to denote such a specific collection of sensitivity distributions, which we term a population.

**B. PoA and Robustness**

For a given routing problem $G \in \mathcal{G}$, we gauge the efficacy of a collection of taxation functions $\tau = \{ \tau_e \}_{e \in E}$ by comparing the total latency of the resulting Nash flow and the total latency associated with the optimal flow and then performing a worst-case analysis over all possible user populations. Let $L^c (G)$ denote the total latency associated with the optimal flow, and $L^c (G, s, \tau)$ denote the total latency of the worst-performing Nash flow resulting from taxation functions $\tau$ and population $s$. The worst-case system cost associated with this specific instance is captured by the PoA, which is of the form

$$\text{PoA} (G, \tau) = \sup_{s \in \delta_G} \left\{ \frac{L^c (G, s, \tau)}{L^c (G)} \right\} \geq 1. \quad (5)$$

In this context, we seek taxation mechanisms that minimize $\text{PoA} (G, \tau)$ for a wide variety of routing games $G$. If a taxation mechanism $\tau$ brings $\text{PoA} (G, \tau)$ close to 1 for many games $G$ (in a sense to be made exact later), this indicates that $\tau$ is robust to mischaracterizations of user sensitivities. Traditionally, the PoA is analyzed in worst case over a given class of games [14]. In our usage, we delay taking the worst case over all networks until specific settings when it is called for. For example, Theorem 1 exhibits a taxation mechanism, which drives the PoA to 1 for every $G$, but the rate at which the PoA approaches 1 may vary from network to network. On the other hand, Theorem 3 provides an expression for the PoA that holds for all parallel networks.

**III. RELATED WORK**

The following is a brief overview of the existing literature on taxation mechanisms in this context. A taxation mechanism simply computes edge tolls as a function of some set of information about the system; here, we focus in particular on the informational dependencies of several well-studied taxation approaches.

1) Omniscient taxation mechanisms: These taxation mechanisms are assumed to have access to complete information regarding the routing game. For edge $e \in G$ and population $s \in \delta_G$, the edge tolling function takes the following form:

$$\tau_e (f_r; G, s).$$

That is, each edge’s taxation function can depend on the entire routing problem $G$ and the population sensitivities $s$. Recent results have identified taxation mechanisms of this form that assign fixed tolls (i.e., for any $e \in G$, $\tau_e (f_r) = q_e$ for some $q_e \geq 0$) that can enforce any feasible flow [20], [21], thus guaranteeing a PoA of 1. However, the robustness of these mechanisms to variations or mischaracterizations of network topology and user sensitivities is heretofore unknown.

2) Network-agnostic taxation mechanisms: This type of taxation mechanism is agnostic to network specifications: each taxation function is derived from locally available information only. Here, a system designer essentially commits to a taxation function for each potential edge $e \in G$, and any network realization $G \in \mathcal{G}$ merely employs a subset of these predefined taxation functions. An edge’s toll cannot depend on any other edge’s cost or location in the network, nor can it depend on the tax sensitivities of the agents.

A commonly studied network-agnostic taxation mechanisms is the marginal-cost (or Pigovian) taxation mechanism $\tau^mc$, which is of the following form: for any $e \in G$ with latency function $\ell_e$, the accompanying taxation function is

$$\tau^mc_e (f_r) = f_e \cdot \frac{d}{df_e} \ell_e (f_r), \quad \forall f_e \geq 0. \quad (6)$$

In [22], it is shown that for any $G \in \mathcal{G}$, we have $L^c (G) = L^mc (G, s, \tau^mc)$, provided that all users have a sensitivity exactly equal to 1. Hence, irrespective of the underlying network structure, a marginal-cost taxation mechanism always ensures the optimality of the resulting Nash flow, provided that all users share a common known sensitivity.

There are many other results in this area; for example, in [27], the authors investigate the PoA of various types of tolling functions with built-in upper bounds. In [28], it is shown that if taxes can be computed in a centralized fashion, any feasible flow can be enforced even if the central planner does not know the network’s latency functions. For affine-cost parallel networks, Christodoulou et al. [29] derive omniscient flow-varying taxation mechanisms for applications where the total traffic rate is unknown. Finally, in [7], the authors show that marginal-cost taxes scaled by $\sqrt{S_L^c S_L}$ do possess a degree of robustness to mischaracterizations of user sensitivities for affine-cost parallel networks.

**IV. UNIVERSAL TAXATION MECHANISM**

In this paper, we prove that network- and sensitivity-agnostic tolls exist, which can drive the PoA to 1 for general networks and latency functions. We term these “universal” because they take the same form and provide the same performance guarantee regardless of which particular routing scenario they are applied to. Using this taxation mechanism, we show in Theorem 1 that for any network, regardless of the network topology, traffic rates $\{r^c\}$, or price-sensitivity functions $\{s^c\}$, the PoA can be made arbitrarily close to 1 with sufficiently large edge tolls, indicating that tolls exist, which are robust to mischaracterizations of all the aforementioned system parameters.

**Theorem 1:** Let $G$ be the set of multi commodity routing games, where $S_L^c \geq S_L > 0$. For any network edge $e \in G$ with convex, nondecreasing, continuously differentiable latency function $\ell_e$, define the universal taxation function on edge $e$ with gain parameter $\kappa \geq 0$ as

$$\tau^uf_e (f_r; \kappa) = \kappa \left( \ell_e (f_r) + f_e \cdot \frac{d}{df_e} \ell_e (f_r) \right). \quad (7)$$
Then, for any routing problem \( G \in \mathcal{G} \), we have
\[
\lim_{\kappa \to -\infty} \text{PoA}(G, \tau^u(\kappa)) = 1.
\] (8)

That is, on any network being used by any population of users, the total latency can be made arbitrarily close to the optimal latency, and each individual link toll is a simple continuous function of that link’s flow. The reason for this is that as \( \kappa \) increases, the original latency function has a smaller and smaller relative effect on the users’ cost functions; in the large-toll limit, the only cost experienced by the users is the tolling function itself, which is specifically designed to induce optimal Nash flows.

Proof: Using a sequence of tolls, we construct a sequence of Nash flows that converges to an optimal flow. Let \( \kappa_n \) be an unbounded increasing sequence of toll coefficients.

For any routing problem \( G \in \mathcal{G} \) and price sensitivities \( s \in \mathcal{S}_G \), let \( f^n = (f^n_p)_{p \in \mathcal{P}} \) denote the Nash flow resulting from the tolling coefficient \( \kappa_n \). For each commodity \( c \), let \( \mathcal{P}^c \subseteq \mathcal{P} \) denote the set of paths that have positive flow in \( f^n \). For any \( p \in \mathcal{P}^c \), there must be some user \( x \in [0, r^c] \) using \( p \) with sensitivity \( s^c \); the cost experienced by this user is given by
\[
J^n_s(f^n) = \sum_{e \in p} \left[ \ell'_e(f^n) + \kappa_n s^c_e \left( \ell_e(f^n) + f^n_e \cdot \frac{d}{df^n_e} \ell_e(f^n) \right) \right].
\]

Define \( \gamma_{n,x} \triangleq \frac{\kappa_n s^c_e}{1 + \kappa_n s^c_e} \). Let \( \ell_e^*(f_n) = f^n_e \cdot \frac{d}{df^n_e} \ell_e(f^n) \); then, for any other path \( p' \in \mathcal{P}^c \setminus p \), user \( x \) must experience a lower cost on \( p \) than on \( p' \), or
\[
\sum_{e \in p} \ell_e^*(f_n) - \sum_{e \in p'} \ell_e^*(f_n) \leq \gamma_{n,x} \sum_{e \in p'} \ell_e^*(f_n) - \sum_{e \in p} \ell_e^*(f_n).
\] (9)

Therefore, for any \( n \geq 1 \), \( f^n \) must satisfy some set of inequalities defined by (9). Note that for all \( c \in \mathcal{C} \) and any \( x \in [0, r^c] \), \( \lim_{\kappa \to -\infty} \gamma_{n,x} = 1 \), so because all the functions in (9) are continuous, \( f^n \) converges to a set of feasible flows that satisfy
\[
\sum_{e \in p} \ell_e^*(f_n) - \sum_{e \in p'} \ell_e^*(f_n) \leq \sum_{e \in p'} \ell_e^*(f_n) - \sum_{e \in p} \ell_e^*(f_n)
\] (10)

for all \( c \), all \( p \in \mathcal{P}^c \), and \( p' \in \mathcal{P}^c \), where \( \mathcal{P}^c \subseteq \mathcal{P} \) is some subset of paths. But inequalities (10) (combined with the feasibility constraints on \( f \)) also specify a Nash flow for \( G \) for a unit-sensitivity population with marginal-cost taxes as specified by (6). Any such Nash flow must be optimal [22], that is, any \( f \in F^w \) is a minimum-latency flow for \( G \). Thus, since \( \mathcal{L}(f) \) is a continuous function of \( f \), then
\[
\lim_{n \to \infty} \mathcal{L}(f^n) = \mathcal{L}^*(G)
\] (11)

obtaining the proof of the theorem.

### A. PoA Bounds for Homogeneous Populations

The result in Theorem 1 is encouraging, since it ensures that no routing game or user population is so pathological that we cannot enforce optimal routing with sufficiently high tolls, but it gives no indication of how high these tolls must be. In our next result in Proposition 2 (which follows from a result in [30]), we state that for homogeneous price-sensitive populations (i.e., all users have the same nonzero price sensitivity), the performance degradation is uniformly bounded in all games by a simple expression.

**Proposition 2:** If all users have (unknown) homogeneous price sensitivity \( s \geq S_L > 0 \), the PoA induced by \( \tau^u(\kappa) \) is given by
\[
\sup_{G \in \mathcal{G}} \text{PoA}(G, \tau^u(\kappa)) \leq \frac{1 + \kappa S_L}{\kappa S_L}.\] (12)

**Proof:** Immediate from [30, Proposition 6.4].

### B. Examples Illustrating Universal Tolls

**Example 1:** Consider the network in Fig. 1. This network has been used to demonstrate dynamic tolling mechanisms [13], and we use variations of it to illustrate the universal tolls of Theorem 1. This network has three commodities, labeled in Fig. 1 as \( (A, B) \) (red), \( (A, E) \) (green), and \( (C, D) \) (blue), with associated traffic rates \( r_1, r_2 \), and \( r_3 \), respectively. Traffic in each commodity has access to all directed paths that connect the respective source and destination; for example, \( (A, B) \) can choose between \( \{e_1\}, \{e_2, e_3, e_4\} \), and \( \{e_5, e_6, e_7, e_8\} \). For a demonstration of universal tolls applied to a specific instance of this network, see Fig. 2. To demonstrate the effects of the universal tolls of Theorem 1, random variations of this network are simulated and the resulting PoA values are plotted in Fig. 3.

![Fig. 1. Base network for Examples 1 and 2. This network has three commodities (i.e., source–destination pairs): (A, B) (red), (A, E) (green), and (C, D) (blue), with associated traffic rates r1, r2, and r3, respectively. Traffic in each commodity has access to all directed paths that connect the respective source and destination; for example, (A, B) can choose between {e1}, {e2, e3, e4}, and {e5, e6, e7, e8}.](image-url)

See Table I.

**Table I.**

<table>
<thead>
<tr>
<th>( e )</th>
<th>( \alpha_e )</th>
<th>( \beta_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>e1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>e2</td>
<td>0.9</td>
<td>0.1</td>
</tr>
<tr>
<td>e3</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>e4</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>e5</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>e6</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>e7</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>e8</td>
<td>0.3</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Random variations of this network are simulated and the resulting PoA values are plotted in Fig. 3.

First, consider an instance of this network in which \( r_1 = r_2 = r_3 = 0.5 \), all traffic in commodity 1 has sensitivity \( s_1 = 100 \), and all traffic in commodities 2 and 3 have sensitivity \( s_2 = s_3 = 3 \). Let the latency functions be given by \( \ell_e(f_e) = a_e f_e^3 + b_e \), with coefficients \( a_e \) and \( b_e \) given in Table I. Quartic latency functions of this form are a stylized form of the well-known Bureau of Public Roads (BPR) latency functions, commonly used to model the congestion characteristics of physical roads [13], [31]. The optimal flow on this network [see Fig. 2(a)] has a total latency of approximately 1.49; the uninfluenced Nash flow [see Fig. 2(b)] has a total latency of approximately 1.955, for an uninfluenced PoA of about 1.31. Applying universal tolls (7) to this network (tolling functions in Table I) results in an improvement in the total latency; Nash flows for \( \kappa = 0.5 \) and \( \kappa = 10 \) are depicted in Figs. 2(c) and (d), respectively. Fig. 2(e) plots the PoA of this specific instance as a function of \( \kappa \). Values of \( \kappa \) as low as 0.1 reduce the PoA from 1.31 to 1.19; when \( \kappa \geq 5 \), the PoA is already below 1.05. For comparison, the worst-case PoA (over all networks) for quartic latency functions is approximately 2.15 [14].
TABLE I
LATENCY AND UNIVERSAL TOLLING FUNCTIONS FOR EXAMPLE 1

<table>
<thead>
<tr>
<th>edge</th>
<th>$\ell_e(f_e) = a_e(f_e)^4 + b_e$</th>
<th>$\tau^u_e(f_e) = \kappa \left(5a_e(f_e)^4 + b_e\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$0.88(f_e)^4 + 0.10$</td>
<td>$\kappa \left(4.40(f_e)^4 + 0.10\right)$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$0.59(f_e)^4 + 0.91$</td>
<td>$\kappa \left(2.95(f_e)^4 + 0.91\right)$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$0.66(f_e)^4 + 0.87$</td>
<td>$\kappa \left(3.30(f_e)^4 + 0.87\right)$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>$0.24(f_e)^4 + 0.88$</td>
<td>$\kappa \left(1.20(f_e)^4 + 0.88\right)$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>$0.57(f_e)^4 + 0.93$</td>
<td>$\kappa \left(2.85(f_e)^4 + 0.93\right)$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>$0.22(f_e)^4 + 0.12$</td>
<td>$\kappa \left(3.10(f_e)^4 + 0.12\right)$</td>
</tr>
<tr>
<td>$e_7$</td>
<td>$0.89(f_e)^4 + 0.34$</td>
<td>$\kappa \left(4.45(f_e)^4 + 0.34\right)$</td>
</tr>
<tr>
<td>$e_8$</td>
<td>$0.93(f_e)^4 + 0.93$</td>
<td>$\kappa \left(4.65(f_e)^4 + 0.93\right)$</td>
</tr>
<tr>
<td>$e_9$</td>
<td>$0.68(f_e)^4 + 0.22$</td>
<td>$\kappa \left(3.40(f_e)^4 + 0.22\right)$</td>
</tr>
<tr>
<td>$e_{10}$</td>
<td>$0.31(f_e)^4 + 0.72$</td>
<td>$\kappa \left(1.55(f_e)^4 + 0.72\right)$</td>
</tr>
<tr>
<td>$e_{11}$</td>
<td>$0.26(f_e)^4 + 0.40$</td>
<td>$\kappa \left(1.30(f_e)^4 + 0.40\right)$</td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>$0.54(f_e)^4 + 0.45$</td>
<td>$\kappa \left(2.70(f_e)^4 + 0.45\right)$</td>
</tr>
<tr>
<td>$e_{13}$</td>
<td>$0.06(f_e)^4 + 0.08$</td>
<td>$\kappa \left(0.30(f_e)^4 + 0.08\right)$</td>
</tr>
</tbody>
</table>

Fig. 2. Specific instance of network from Fig. 1 for Example 1. Commodity traffic rates are $r_1 = r_2 = r_3 = 0.5$, and each commodity is assigned homogeneous sensitivity values of $s_1 = 100$ and $s_2 = s_3 = 0.1$. Latency functions of $\ell_e(f_e) = a_e(f_e)^4 + b_e$ were assigned to the network, with values as shown in Table I. The optimal flow on the network was computed, as well as Nash flows resulting from universal tolls (7) for several values of parameter $\kappa$. In (a)–(d), a colored edge indicates that traffic from the commodity corresponding to that color is using that edge; a thin black edge indicates no flow. Part (e) plots the simulated PoA on this network, for this population, as a function of tolling parameter $\kappa$.

Example 2: The PoA curve in Fig. 2(e) is specific to one particular routing problem and user population; different networks and populations have different curves. To study the effect of universal tolls on more than this single instance, networks were generated by randomly deleting edges from the network in Fig. 1 (while requiring that each commodity has a feasible S-D path and that at least one commodity has more than one such path). BPR latency functions were chosen of the form $\ell_e(f_e) = a_e(f_e)^4 + b_e$, where the $a_e$ and $b_e$ coefficients were chosen independently and uniformly at random from the interval $[0, 1]$. Each network was simulated with several homogeneous and heterogeneous user populations and several different traffic rates, for a total of 2,314 individual routing problems simulated. The ratio of Nash total latency to optimal total latency (instance-specific PoA) was computed on each of these 2,314 routing problems in response to universal tolls (7) for a specific value of $\kappa$. The solid blue, dashed green, and dotted red lines represent, respectively, the maximum, 99th percentile, and 75th percentile of PoA values for each corresponding value of $\kappa$. Note in particular that the 75th percentile line is less than 1.01 for all $\kappa \geq 0.5$, suggesting that relatively low tolls may suffice for many networks.

V. THEOREM 3: OPTIMAL BOUNDED TOLLS

Of course, it may be impractical or politically infeasible to charge extremely high tolls. For example, if network demand is elastic, very large tolls could induce some users to avoid travel altogether. Therefore, in Theorem 3, we analyze the effect of an upper bound on the allowable tolling functions. For simplicity, we focus on parallel networks, which have been used to model problems such as scheduling small jobs on machines [32]. For parallel networks with affine cost functions in which every edge has positive flow in an untolled Nash flow, we explicitly derive the optimal bounded taxation mechanism and then provide an expression for the PoA. These optimal tolls are simple affine functions of flow, and the PoA is strictly decreasing in the upper bound. Formally, we say a taxation mechanism is bounded if all its taxation functions respect some upper bound.

Definition 1: Taxation mechanism $\tau$ is bounded by $T$ on a class of routing problems $\mathcal{G}$ if for every edge $e \in \mathcal{G}$, $\tau$ assigns a (possibly flow-varying) tolling function that satisfies

$$\tau_e : [0, 1] \rightarrow [0, T].$$

$\mathcal{F}(T, \mathcal{G})$ denotes the set of mechanisms bounded by $T$ on $\mathcal{G}$.

For the following results, let $\mathcal{G}^e \subseteq \mathcal{G}$ represent the class of all single-commodity parallel-link routing problems with affine latency functions. That is, for all $e \in \mathcal{G}^e$, the latency function satisfies

$$\ell_e(f_e) = a_e f_e + b_e.$$
where \( a_e \geq 0 \) and \( b_e \geq 0 \) are edge-specific constants. “Single commodity” implies that all traffic has access to all network edges. Furthermore, we assume that every edge has positive flow in an untopled Nash flow.\(^3\) In order to meaningfully discuss uniform toll bounds on a broad class of networks, it is necessary to describe classes of networks with bounded latency functions. To this end, we define \( G(\bar{a}, \bar{b}) \subseteq G^p \) as the set of parallel affine-cost networks such that for every \( e \in G(\bar{a}, \bar{b}) \), the latency function coefficients satisfy \( a_e \leq \bar{a} \) and \( b_e \leq \bar{b} \). Note that \( \bar{a} \) and \( \bar{b} \) represent the maximum possible congestibility and free-flow time, respectively; estimates of these quantities should be available because they are functions of physical parameters such as distance and road width.

**Definition 2:** For every edge \( e \in G \) with the latency function \( \ell_e \), a network-agnostic taxation mechanism is a mapping \( \tau^n : [0, 1] \times \{ \ell_e \} \rightarrow \{ \tau_e \} \) that assigns the following flow-dependent taxation function to edge \( e \):

\[
\tau_e(f_e) = \tau^n(f_e; \ell_e)
\]

where \( \tau^n(f_e, \ell_e) \) satisfies the following additivity condition:\(^4\) for all \( e, e' \in G \) and \( f \in [0, 1] \), we have

\[
\tau^n(f_e; \ell_e + \ell_{e'}) = \tau^n(f_e; \ell_e) + \tau^n(f_e; \ell_{e'}).\]

Thus, both marginal-cost tolls (6) and universal tolls (7) are network agnostic according to Definition 2.

Our goal is to derive the bounded network-agnostic taxation mechanism that minimizes the worst-case selfish routing on \( G^p \).

We define the PoA with respect to class of problems \( G \) and bound \( T \) as the best PoA we can achieve on \( G \) with a taxation mechanism bounded by \( T \):

\[
\text{PoA}_T(G) \triangleq \inf_{\tau \in \mathcal{F}(T, G)} \left\{ \sup_{G \in \mathcal{G}} \text{PoA}(G, \tau) \right\}.
\]

**Theorem 3:** Let \( G(\bar{a}, \bar{b}) \subseteq G^p \) be some subset of parallel affine-cost networks with finite \( \bar{a} \) and \( \bar{b} \). For any toll bound \( T \) and \( S_U \geq S_L > 0 \), define the set of universal parameters by the tuple \( T' = (S_L, S_U, \bar{a}, \bar{b}) \). Then, there exist functions \( \kappa_1(U_T) \) and \( \kappa_2(U_T) \) such that the optimal network-agnostic taxation mechanism bounded by \( T \) on \( G(\bar{a}, \bar{b}) \) assigns tolling functions

\[
\tau_e(f_e) = \kappa_1(U_T) a_e f_e + \kappa_2(U_T) b_e.
\]

Furthermore, \( \text{PoA}_T(G(\bar{a}, \bar{b})) \) is given by the following:

\[
\frac{4}{3} \left( 1 - \frac{\kappa_1(U_T) S_L + \kappa_2(U_T) S_U}{1 + \kappa_1(U_T) S_L} \right), \quad \text{if } \kappa_1(U_T) \leq \frac{1}{\sqrt{S_L S_U}};
\]

\[
\frac{4}{3} \left( \frac{1 + \kappa_1(U_T) S_L}{1 + 2 \kappa_1(U_T) S_L + \frac{1}{S_L}} \right), \quad \text{if } \kappa_1(U_T) \geq \frac{1}{\sqrt{S_L S_U}}.
\]

See Fig. 4 for a comparison of the PoA afforded by Theorems 1 and 3. Note that the tolls of Theorem 3 incentivize considerably lower system costs than those of Theorem 1; this is due to the fact that Theorem 3 is optimized for a smaller class of networks.

For the reader’s convenience, we include a closed-form expression for \( \kappa_1(\cdot) \) in the Appendix as (43) in Fig. 7, and for \( \kappa_2(\cdot) \) in the proof of Theorem 3 as (27). It is evident from these expressions that \( \kappa_1(\cdot) \) and \( \kappa_2(\cdot) \) are both nondecreasing and unbounded in \( T \); among other things, this implies that \( \lim_{T \rightarrow \infty} \text{PoA}_T(G(\bar{a}, \bar{b})) = 1 \).

We now proceed with the proof of Theorem 3, which relies on two supporting lemmas. For our first milestone, we restrict attention to simple affine taxation functions:

**Lemma 2.1:** Let \( \tau^A(\kappa_1, \kappa_2) \) denote an affine taxation mechanism that assigns tolling functions \( \tau_e(f_e) = \kappa_1 a_e f_e + \kappa_2 b_e \).

For any \( \kappa_{\text{max}} \geq 0 \), the optimal coefficients \( \kappa_1^* \) and \( \kappa_2^* \) satisfying

\[
(\kappa_1^*, \kappa_2^*) \in \arg \min_{\kappa_1, \kappa_2 \leq \kappa_{\text{max}}} \left\{ \sup_{G \in \mathcal{G}} \text{PoA}(G, \tau^A(\kappa_1, \kappa_2)) \right\}
\]

are given by

\[
\kappa_1^* = \kappa_{\text{max}},
\]

\[
\kappa_2^* = \max \left\{ 0, \frac{\kappa_{\text{max}}^2 S_L S_U - 1}{S_L + S_U + 2 \kappa_{\text{max}} S_L S_U} \right\}.
\]

Furthermore, for any \( G \in \mathcal{G}^p \), \( \text{PoA}_T(G, \tau^A(\kappa_1^*, \kappa_2^*)) \) is upper bounded by the following expression:

\[
\frac{4}{3} \left( 1 - \frac{\kappa_{\text{max}} S_L}{1 + \kappa_{\text{max}} S_L} \right), \quad \text{if } \kappa_{\text{max}} \leq \frac{1}{\sqrt{S_L S_U}};
\]

\[
\frac{4}{3} \left( \frac{1 + \kappa_{\text{max}} S_L}{1 + 2 \kappa_{\text{max}} S_L + \frac{1}{S_L}} \right), \quad \text{if } \kappa_{\text{max}} \geq \frac{1}{\sqrt{S_L S_U}}.
\]

See the Appendix for the proof of Lemma 2.1.

Next, in Lemma 2.2, we investigate the possibility that some other taxation mechanism could perform better than the affine \( \tau^A(\kappa_1^*, \kappa_2^*) \) while still respecting the bound \( T \). To that end, we assume that some arbitrary taxation mechanism outperforms affine tolls and deduce various properties of these hypothetical tolls. We show that this hypothetical “better” taxation mechanism must universally charge higher tolls than our optimal affine tolls.
Lemma 2.2: Let $\tau^*$ be any network-agnostic taxation mechanism such that for $\kappa_{\max} \geq 0$, we have
\[
\sup_{G \in \mathcal{G}} \text{PoA}(G^p, \tau^*) < \sup_{G \in \mathcal{G}} \text{PoA}(G^p, \tau^A(\kappa_1^*, \kappa_2^*)) .
\] (24)
Then, $\tau^*$ must charge strictly higher tolls than $\tau^A(\kappa_1^*, \kappa_2^*)$ on every edge in every network
\[
\forall e \in G^p, \forall f_e \in (0, 1], \quad \tau^*_e(f_e) > \tau^A_e(f_e). \tag{25}
\]

The proof of Lemma 2.2 appears in the Appendix.

Proof of Theorem 3: For any nonnegative $\kappa_1$ and $\kappa_2$, $\tau^A(\kappa_1, \kappa_2)$ is tightly bounded by $(\kappa_1 \bar{a} + \kappa_2 \bar{b})$ on $G(\bar{a}, \bar{b})$. Note that $\kappa_1^*$ and $\kappa_2^*$ as defined in Lemma 2.1, $(\kappa_1^* \bar{a} + \kappa_2^* \bar{b})$ is a strictly increasing continuous function of $\kappa_{\max}$. Thus, for any $T \geq 0$, there is a unique $\kappa_{\max}^* \geq 0$, for which $\tau^A(\kappa_1^*, \kappa_2^*)$ is tightly bounded by $T^* = T$ on $G(\bar{a}, \bar{b})$. We define the function $\kappa_1^*(U_T)$ as the maximal $\kappa_{\max}^*$ for any $T \geq 0$, given $S_L, S_U, \bar{a}, \bar{b}$ and $T$. That is, $\kappa_1^*(U_T)\bar{a}$ is implicitly defined as the unique function satisfying
\[
\kappa_1^*(U_T)\bar{a} + \max \left\{ 0, \frac{(\kappa_1^2(U_T)S_L S_U - 1) \bar{b}}{S_L + S_U + 2(\kappa_1(U_T)S_L S_U)} \right\} = T. \tag{26}
\]
For completeness, in the Appendix, we include a closed-form expression for $\kappa_1^*(U_T)$ as (43). We define $\kappa_2^*(U_T)$ as
\[
\kappa_2^*(U_T) = \max \left\{ 0, \frac{(\kappa_2^2(U_T)S_L S_U - 1) \bar{b}}{S_L + S_U + 2(\kappa_2(U_T)S_L S_U)} \right\}. \tag{27}
\]

Let $e' \in G$ be an edge with the latency function $\ell_e^*(f_{e'}) = \bar{a} f_{e'} + \bar{b}$. By construction, the tolling function assigned by $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$ to $e'$ satisfies bound $T$ with equality: $\tau^*_e(1) = T$.

Now, let $\tau^*$ be any taxation mechanism with a strictly lower PoA than $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$. By Lemma 2.2, $\tau^*$ assigns higher tolling functions than $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$ on every edge for every flow rate. In particular, on edge $e'$, $\tau^*_e(1) > \tau^*_e(1) = T$, violating bound $T$ and proving the optimality of $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$ over the space of all network-agnostic taxation mechanisms bounded by $T$. By substituting $\kappa_1^*(U_T)$ for $\kappa_{\max}^*$ in expression (23), we obtain the complete PoA expression (19).

It may be helpful to note that the crucial point in Theorem 3 is that the upper bound $T$ allows us to compute a maximum tolling coefficient $\kappa_{\max}$; it is this $\kappa_{\max}$ that enters the PoA expression in (19). Thus, an alternative formulation of boundedness is possible, which simply specifies a $\kappa_{\max}$ and dispenses with specifying $T$, $\bar{a}$, and $\bar{b}$. This formulation represents a relative boundedness in which tolls cannot be too much larger than realized latency function parameters.

VI. Negative Results for Fixed Tolls

Theorem 3 showed that simple affine tolling functions are sufficient to achieve the best-possible PoA for network-agnostic bounded taxation mechanisms. It is natural to ask what guarantees are possible for an even simpler class of taxation functions, the constant functions. There are practical benefits to such fixed tolls, foremost among which is the simplicity and predictability they offer to network users.

It has long been known that flow-varying tolls are sufficient to optimize network routing in cases when the network topology is unknown [22]. We ask here if fixed tolls can provide the same guarantee, i.e., we ask if (strictly) flow-varying tolls are also necessary to optimize routing in these settings. In Theorem 4, we prove this necessity, which immediately implies that the PoA of network-agnostic fixed tolls is bounded away from 1.

Theorem 4: If for every $G \in \mathcal{G}$ and unit-sensitivity homogeneous population $s$, network-agnostic taxation mechanism $\tau$ satisfies
\[
L^*(G, s, \tau) = L^*(G) \tag{28}
\]
then it must be the case that $\tau$ assigns strictly flow-varying taxation functions to some network edges.

Proof: We prove Theorem 4 by contradiction. Let $\tau^{na}$ be a network-agnostic fixed-tolling mechanism for which $L^{na}(G, s, \tau^{na}) = L^*(G)$, that is, it is a mapping from latency functions to nonnegative constant taxation functions that enforces optimal routing on every network. Consider the two-path network shown in Fig. 5(a); denote this network $G_n$. The upper path is composed of $n$ copies of the same link in series; network agnosticity requires that $\tau^{na}$ charges the same toll to every copy of that link. For a total traffic mass of $r$, the optimal routing profile for this network is $f_1^* = b/2$ and $f_2^* = r - b/2$. For a unit-sensitivity homogeneous population, optimal fixed tolls $\tau_1$ and $\tau_2$ must satisfy
\[
\tau_2 = n \tau_1 - b/2. \tag{29}
\]
Since these tolls are network agnostic, $\tau_1$ cannot be a function of $b$, so there exists some universal constant $\beta > 0$ for which $\tau_1 = \beta$ and $\tau_2 = n \beta - b/2$. It is straightforward to show that for any $n$ and any choice of $\beta$, these tolls induce optimal routing on the network for a unit-sensitivity homogeneous user population. That is, $L^{na}(G_n, s, \tau^{na}) = L^*(G_n)$.

Our hope is that these tolling functions would optimize routing when applied to any network, i.e., we could apply $\tau_1 = \beta$ to any edge with latency function $\ell_e(f_e) = f_e$, and $\tau_2$ to any edge with latency function $b$ and still get optimal performance. To test this, we apply the same tolls to the network in Fig. 5(b), which we denote $G_1$. Here, we find that $\tau_2 = n \beta - b/2$ is now much too high; if the total traffic rate is high enough, these tolls induce a flow with $f_1 = \beta(n-1) + b/2$ and $f_2 = 0$, even though the optimal flow has $f_1 = b/2$. This allows us to compute a lower bound on the PoA for these tolling functions
\[
\frac{L^{na}(G_1, s, \tau^{na})}{L^*(G_1)} \geq \frac{(\beta(n-1) + \frac{b}{2})^2}{b(1 + \frac{b}{2})} \tag{30}
\]
which is unbounded in both $n$ and $\beta$, generating a contradiction to our hypothesis that for all $G$, $\mathcal{L}^{nf}(G, s, \tau^{wh}) = \mathcal{L}(G)$.

In light of this negative result, in Theorem 5, we ask what guarantees are possible with fixed tolls if we know the network structure but do not know the user sensitivities; refer to the last row of Table II for a quick summary of the setting we investigate here. Since we are allowing these fixed tolls to depend on the network structure (e.g., the number of edges in the network), we denote such taxation functions by $\tau^{ft}(G) = \{\tau^{ft}_e(G)\}_{e \in G}$. The following theorem demonstrates that any network-dependent fixed-toll taxation mechanism generally provides poor performance guarantees when compared with the optimal bounded taxation mechanism from Theorem 3.

**Theorem 5:** Consider any network-dependent fixed-toll taxation mechanism $\tau^{ft}$. For any network $G \in \mathcal{G}^p$, we have

$$\sup_{s \in S} \mathcal{L}^{nf}(G, s, \tau^{ft}(G)) \geq \sup_{s \in S} \mathcal{L}^{nf}(G, s, \tau^{A}(1/S_U, 0))$$

with affine tolls $\tau^{A}(\cdot)$ as defined in Lemma 2.1. Thus

$$\sup_{G \in \mathcal{G}} \mathcal{P}_A(G, \tau^{ft}) \geq \sup_{G \in \mathcal{G}^p} \mathcal{P}_A(G, \tau^{A}(1/S_U, 0)) = \frac{4}{3} \left(1 - \frac{S_L/S_U}{(1 + S_L/S_U)^2}\right).$$

We point out that the right-hand side of (32) represents the PoA due to network-agnostic affine tolls for a very low toll upper bound. For example, in the canonical Pigou network depicted in Fig. 4, if $S_U = 10$, affine tolls prescribed by $\tau^{A}(1/S_U, 0)$ imply a toll upper bound of just 0.1. As shown in Fig. 4, the PoA for optimal affine tolls is steeply decreasing in the toll upper bound, so a designer wishing to exploit the simplicity of fixed tolls may need to accept lower performance guarantees as a result.

Furthermore, it is important to note that Theorem 5 shows that $\tau^{A}$, a network-agnostic tolling mechanism, provides better performance guarantees (even for moderately low tolls) than $\tau^{ft}$, a network-dependent tolling mechanism. This shows the power of Theorem 3’s taxation mechanism: given less information, it performs better than any fixed-toll taxation mechanism.

See Fig. 6 for a comparison of the PoA afforded by Theorems 3 and 5, and note that fixed tolls only outperform flow-varying affine tolls when both uncertainty and the toll upper bound are low. In all other situations, optimal affine tolls provide better performance guarantees.

The proof of Theorem 5 first considers homogeneous sensitivity distributions and then extends to heterogeneous. We write $f^{ft}(G, s, \tau)$ and $\mathcal{L}^{nf}(G, s, \tau)$ to denote a Nash flow and its associated total latency induced by fixed tolls $\tau \in \mathbb{R}_+^n$ on network $G$, with homogeneous sensitivity $s \in [S_L, S_U]$. Similarly, we write the total latency of a Nash flow resulting from affine tolls $\tau^A(\kappa_1, \kappa_2)$ as $\mathcal{L}^{nf}(G, s, \tau^A(\kappa_1, \kappa_2))$.

Define the optimal fixed tolls $\tau^{*}$ as

$$\tau^{*} \in \arg \min_{\tau \in \mathbb{R}_+^n} \max_{s \in [S_L, S_U]} \mathcal{L}^{nf}(G, s, \tau).$$

That is, $\tau^{*}$ is in the set of edge tolls that minimize the total latency for the worst possible user sensitivity.

In Lemma 5.1, we see that there is a curious relationship between the total latencies of Nash flows resulting from fixed tolls and those resulting from affine tolls $\tau^A(1/S_U, 0)$. That is, the optimal fixed tolls guarantee the same worst-case performance as affine tolls with extremely low coefficients.

**Lemma 5.1:** For any $G \in \mathcal{G}^p$, a homogeneous population, the worst-case total latency resulting from the optimal fixed tolls $\tau^{*}$ is equal to the worst-case total latency resulting...
from $\tau(1/S_1, 0)$:

$$\max_{s \in [S_1, S_U]} L^U(G, s, \tau^*) = \max_{s \in [S_1, S_U]} L^U(G, s, \tau^A(1/S_1, 0)).$$  

(34)

The proof of Lemma 5.1 appears in the Appendix.

Proof of Theorem 5: Since the set of homogeneous populations is a strict subset of the set of heterogeneous ones, we can only make things worse by extending from homogeneous to heterogeneous populations, so the bound in (32) must hold. The expression in (32) is obtained by substituting $1/S_1$ in for $\kappa_{max}$ in the first part of expression (23).

\section{VII. Conclusion}

In this paper, we have explored several avenues for influencing social behavior when aspects of the underlying system are uncertain. Table II shows our results in context with previous results on this topic, illustrating the informational requirements and sophistication required of each taxation mechanism.

Avenues for future work include incorporating our results into recent studies on the PoA for unknown or varying traffic rates [15], [16]. How would knowledge of traffic rate factor into our taxation designs? Furthermore, in practical problems, it may be that not every edge is available for taxation; this prompts the question: Which edges are best suited for taxes if other parameters are uncertain?

\section{Appendix}

Proofs of Supporting Lemmas

To prove Lemma 2.1, we analytically relate the Nash flows induced by affine tolls with coefficients $\kappa_1$ and $\kappa_2$ to the Nash flows induced by marginal-cost tolls scaled by $\kappa_1$ for some other sensitivity distribution $s'$. We can then use known analytical techniques for scaled marginal-cost tolls to derive the optimal $\kappa_1$ and $\kappa_2$. We make use of the following theorem.

Theorem 6 (See [7]): For any routing problem $G \in \mathcal{G}^p$ satisfying the assumptions of Theorem 3, the scaled marginal-cost taxation mechanism $\tau^{\text{smc}}(\kappa)$ assigns the following tolls to any edge $e \in E$ for $\kappa \geq 0$:

$$\tau_e^{\text{smc}}(f_e) = \kappa a_e f_e.$$  

(35)

The unique cost-minimizing marginal-cost toll scalar is

$$\kappa^* = \frac{1}{\sqrt{S_L S_U}} = \arg\min_{\kappa \geq 0} \{\text{PoA}(G, \tau^{\text{smc}}(\kappa))\}. $$  

(36)

Finally, for any $G \in \mathcal{G}^p$, for $q = S_L / S_U$, the PoA resulting from the optimal scaled marginal-cost taxation mechanism is

$$\text{PoA}(G, \tau^{\text{smc}}(\kappa^*)) \leq \frac{4}{3} \left(1 - \frac{\sqrt{q}}{1 + \sqrt{q}}\right).$$  

(37)

Proof of Lemma 2.1: Let $G \in \mathcal{G}^p$ and $\kappa_1 \geq \kappa_2 \geq 0.5$ For user $x \in [0, 1]$ with sensitivity $s_x \in [S_1, S_U]$, the cost of edge $e \in G$ given flow $f$ under affine tolls is given by

$$J_x(f) = (1 + \kappa_1 s_x) a_e f_e + (1 + \kappa_2 s_x) b_e.$$  

By a series of algebraic manipulations, we may combine (38) and (39) to obtain

$$J'_x(f) = (1 + \kappa_1 s'_x a_e f_e + b_e)$$  

(40)

which is simply the cost resulting from scaled marginal-cost tolls (35). Thus, for any sensitivity distribution $s$, we may model a Nash flow resulting from affine tolls with coefficients $\kappa_1$ and $\kappa_2$ as a Nash flow for sensitivity distribution $s'$ resulting from scaled marginal-cost tolls with $\kappa = \kappa_1$. Thus, by Theorem 6, assuming first that $\kappa_{max}$ is sufficiently high, our optimal choice of $\kappa_1$ is that which satisfies

$$\kappa_1 = \frac{1}{\sqrt{S_L S_U}}$$  

(41)

where $S'_L$ and $S'_U$ are computed according to (39).

Combining (39) and (41) yields the following characterization of the optimal $\kappa_2$ with respect to $\kappa_1$, for $\kappa_{max} \geq (S_1 S_U)^{-1/2}$:

$$\kappa_2 = \frac{\sqrt{S_L S_U} - 1}{S_L + S_U + 2 S_1 S_2 S'_U}. $$  

(42)

Evaluating (37) at $q = S'_L / S'_U$ verifies the second part of (23) as the correct expression for $\text{PoA}(G, \tau^A(\kappa^*_1, \kappa^*_2))$ when $\kappa_{max}$ is large.

Consider the case when $\kappa_{max} < (S_1 S_U)^{-1/2}$. Now, (42) would prescribe a negative value for $\kappa_2$, so the optimal choice is to let $\kappa_2$ saturate at 0. Now, we are precisely applying scaled marginal-cost tolls with $\kappa = \kappa_1$, so we apply the fact shown in [7, Lemma 1.2] that on this class of networks, if $\kappa \leq (S_1 S_U)^{-1/2}$, the worst-case total latency of a Nash flow always occurs for the extreme low-sensitivity homogeneous sensitivity distribution given by $s_x \equiv S_L$ for all $x \in [0, 1]$.

\footnote{Here, the requirement that $\kappa_1 \geq \kappa_2$ is without loss of generality; later analysis shows that $\kappa_2 > \kappa_1$ would always result in a Nash flow with higher congestion than the untolled case.}
The total latency of a Nash flow for a homogeneous population with sensitivity $S_L$ is given in [7, eq. (35)] as

$$L^\text{nl}(G, S_L, \kappa) = L_R - \frac{\kappa S_L}{(1 + \kappa S_L)^2} \Theta$$

(44)

where $L_R$ and $\Theta$ are positive constants depending only on $G$, satisfying $\Theta \leq L_R$. It is easy to verify that the above expression is minimized on a subset of $[0, (S_L S_U)^{-1/2}]$ by maximizing $\kappa$, and using the fact that $\Theta \leq L_R$, we may verify that the PoA for $\kappa_{\text{max}} < (S_L S_U)^{-1/2}$ is given by the first part of (23), completing the proof of Lemma 2.1.

Proof of Lemma 2.2: Here, we derive properties of any taxation mechanism that outperforms $\tau^A(\kappa_1^*, \kappa_2^*)$. We define the set of routing problems $G^0$ as follows: $G \in G^0$ is a parallel network consisting of two edges, with $\ell_1(f_1) = c f_1$ and $\ell_2(f_2) = c$.

Let $G \in G^0$. For any $c$, the optimal flow on $G$ is $(f_1^*, f_2^*) = (1/2, 1/2)$ and the optimal total latency is $L^\text{c}(G) = 3c/4$, but the untolled Nash flow has a total latency of $L^\text{nl}(G, s, 0) = c$, so the untolled PoA is $4/3$. It is straightforward to show furthermore that if $S_U > S_L \geq 0$, for any $\kappa_{\text{max}} > 0$, this network constitutes a worst-case example and the PoA bound of this particular network is tight, i.e., it equals the expression given in (23): PoA$(G, \tau^A(\kappa_1^*, \kappa_2^*)) = \sup_{G \in G^0} \text{PoA}(G, \tau^A(\kappa_1^*, \kappa_2^*))$.

Thus, if our hypothetical $\tau^*$ outperforms $\tau^A$ in general, it must specifically outperform $\tau^A$ on any network $G \in G^0$, or

$$\text{PoA}(G, \tau^*) < \text{PoA}(G, \tau^A(\kappa_1^*, \kappa_2^*)) \ .$$

(45)

Now, we investigate the performance of the hypothetical tolling mechanism $\tau^*$ on networks in $G^0$. Given a network $G \in G^0$, $\tau^*$ assigns edge tolling functions $\tau_1^*(f_1)$ and $\tau_2^*(f_2)$.

Recall that since $\tau^*$ is network agnostic, there is some function $\tau^*(f; a, b)$ such that an edge $e \in E$ with latency function $\ell_e(f) = a f e + b e$ is assigned tolling function $\tau^*(f; a e, b e)$. By analyzing networks in $G^0$, we can deduce properties of the function with the second and third arguments set to 0, since $\tau_1^*(f_1) = \tau^*(f_1; c, 0)$ and $\tau_2^*(f_2) = \tau^*(f_2; 0, c)$.

Now, we show that $\tau^*$ must assign higher tolls than $\tau^A(\kappa_1^*, \kappa_2^*)$. Let $S_U > S_L$. By design, the worst-case Nash flows resulting from $\tau^A(\kappa_1^*, \kappa_2^*)$ occur for homogeneous populations with $s = S_L$ and $s = S_U$. Since any network $G \in G^0$ has only two links, we can characterize a Nash flow simply by the flow on edge 1; accordingly, let $f_L(c)$ denote the flow as a function of $c$ on edge 1 in the Nash flow resulting from sensitivity distribution $s = S_L$, and $f_H(c)$ the corresponding edge 1 flow for $s = S_U$. These flows are the solutions to the following equations:

$$c f_L(c) (1 + \kappa_1^* S_L) = c (1 + \kappa_2^* S_L) \quad (46)$$

and

$$c f_H(c) (1 + \kappa_1^* S_U) = c (1 + \kappa_2^* S_U) \ .$$

(47)

Summing (46) and (47) yields

$$\kappa_1^*(f_L(c) - f_H(c)) = \frac{f_H(c)}{S_U} - \frac{f_L(c)}{S_L} + \frac{1}{S_L} - \frac{1}{S_U} .$$

(48)

It is always true that $f_H(c) < f_L(c)$. By design, $\mathcal{L}(f_L(c)) = \mathcal{L}(f_H(c))$. Note that $\mathcal{L}$ is simply a concave-up parabola in the flow on edge 1.

Now, let $f_L^*(c)$ and $f_H^*(c)$ be defined as the Nash flows resulting from $\tau^*$ for a given value of $c$, i.e., the solutions to

$$c f_L^*(c) + \tau_1^*(f_L^*(c)) S_L = c + \tau_1^*(1 - f_L^*(c)) S_L \quad (49)$$

and

$$c f_H^*(c) + \tau_1^*(f_H^*(c)) S_U = c + \tau_2^*(1 - f_H^*(c)) S_U .$$

(50)

Since $\tau^*$ guarantees better performance than $\tau^A(\kappa_1^*, \kappa_2^*)$, it must do so in particular for these homogeneous sensitivity distributions $s = S_L$ and $s = S_U$. Since $\mathcal{L}$ is a parabola, this means that for any $c$, $f_L^*(c) < f_H^*(c) < f_L^*(c)$.

Define the nondecreasing function $\Delta^*(f) = \tau_2^*(f) - \tau_1^*(1 - f)$ (which is implicitly also a function of $c$), so (49) and (50) can be combined and rearranged to show

$$\Delta^*(f_L^*(c)) - \Delta^*(f_H^*(c)) > c \left[ \frac{f_H(c)}{S_U} - \frac{f_L(c)}{S_L} + \frac{1}{S_L} - \frac{1}{S_U} \right]$$

$$= \kappa_1^* (f_L(c) - f_H(c)) .$$

(51)

The above inequality can be further loosened by replacing $f_L^*(c)$ with $f_L(c)$ and $f_H^*(c)$ with $f_H(c)$, and substituting from (48) and rearranging, we finally obtain

$$\frac{\Delta^*(f_L(c)) - \Delta^*(f_H(c))}{f_L(c) - f_H(c)} > \kappa_1^* c .$$

(52)

Since this must be true for any $c > 0$, the average slope of $\Delta^*(f)$ must be greater than $\kappa_1^* c$ for all $f > 0$. Since $\tau_2^*(f) \geq 0$, this implies that $\tau_1^*(f) > \kappa_1^* c f$ for all $f > 0$, or that

$$\tau^*(f; a, 0) > \tau^A(f; a, 0)$$

(53)

for all positive $f$ and $a$.

Now, consider the following rearrangement of (50):

$$\tau_2^*(1 - f_H(c)) = [c f_H(c) + \tau_1^*(f_H(c)) - c S_U] \cdot \frac{1}{S_U}$$

$$> c [(1 + \kappa_2^* S_U) f_H(c) - 1] \cdot \frac{1}{S_U}$$

$$= \kappa_2^* c S_U = \tau_2^*(f) .$$

(54)

This implies that $\tau_2^*(f) > \kappa_2^* c$ for all $f > 0$, or that

$$\tau^*(f; 0, b) > \tau^A(f; 0, b)$$

(55)

for all positive $f$ and $b$.

Finally, note that the additivity assumption of Definition 2 implies that $\tau^*(f; a, b)$ is additive in its second and third arguments. That is, we may add inequalities (53) and (55) to arguments. That is, we may add inequalities (53) and (55) to arguments. That is, we may add inequalities (53) and (55) to arguments. That is, we may add inequalities (53) and (55) to arguments. That is, we may add inequalities (53) and (55) to arguments.
Claim 5.1.1: A Nash flow on \( G \) for sensitivity \( s \in S_1 \) and fixed tolls \( \tau \in \mathbb{R}^n \) that has positive traffic on all links can be described by the following linear function:

\[
f^\tau(G, s, \tau) = R + H(b + s\tau)
\]

where \( R \in \mathbb{R}^n \) and \( H \in \mathbb{R}^{n \times n} \) are constant matrices depending only on \( G \). The total latency of this flow is given by the following convex quadratic in \( \tau \):

\[
\mathcal{L}^\tau(G, s, \tau) = L_R + s\tau^TH^T(2AH + I)b + s^2\tau^TH^TAH\tau.
\]

Proof: Since all users share the same sensitivity, all links have equal cost to all agents in a Nash flow, so when all network edges have positive flow, for any \( e_i \), \( e_j \in E \), we have

\[
a_i f_i + b_i + s\tau_i = a_j f_j + b_j + s\tau_j.
\]

Similar to the approach in [7, proof of Lemma 1.2], a Nash flow \( f^\tau(G, s, \tau) \) is a solution to the linear system

\[
\begin{bmatrix}
a_1 & -a_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
f \\
p \\
\tau \\
x
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
-1 & 1 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix} \begin{bmatrix}
(b + s\tau)
\end{bmatrix}.
\]

\[\tag{59}\]

\( P \) is invertible, so letting \( H = P^{-1}X \) and \( R = P^{-1}r \), a Nash flow is given by the linear equation (57).

The following observations will be helpful to our proof.

Observation 5.1: The matrices \( H \) and \( R \) possess the following properties for any \( G \in \mathcal{G} \):

\[
1^T Hb = 0^T
\]

\[
1^T R = 1
\]

\[
AR \in \text{sp}\{1\}
\]

\[
b^T H^T AHB = -M^T b.
\]

Finally, every column of \( (AH + I) \) is in \( \text{sp}\{1\} \).

These facts follow algebraically from the fact that by definition, \( f^\tau(G, s, \tau) \) satisfies (59). Substituting (57) into (1) and simplifying using the facts in Observation 5.1 yields (58). \( \square \)

Next, we establish a necessary condition for a set of fixed tolls to be optimal in the sense of (33).

Claim 5.1.2: Fixed tolls \( \tau^* \) satisfying (33) must also satisfy

\[
H \left( \tau^* + \frac{b}{S_L + S_U} \right) = 0.
\]

Proof: By (58), the total latency due to fixed tolls is a convex parabola in \( s \), so for any \( \tau \), the maximum total latency on \([S_L, S_U]\) occurs at either \( S_L \) or \( S_U \). Since \( \mathcal{L}^\tau(G, s, \tau) \) is continuous and convex in \( \tau \), this means that \( \tau^* \) must satisfy

\[
\mathcal{L}^\tau(G, S_L, \tau^*) = \mathcal{L}^\tau(G, S_U, \tau^*).
\]

Thus, for any optimal fixed tolls \( \tau^* \), \( \mathcal{L}^\tau(G, s, \tau^*) \) is a parabola centered at \( s = \frac{S_L + S_U}{2} \); \( \arg\min_{s \in [S_L, S_U]} \mathcal{L}^\tau(G, s, \tau^*) = (S_L + S_U)/2. \)

\[\tag{66}\]

Our goal is to find the parabola with minimum as in (66), which minimizes the values in (65).

Equation (58) implies that for all \( \tau, \tau' \in \mathbb{R}^n \), \( \mathcal{L}^\tau(G, 0, \tau) = \mathcal{L}^\tau(G, 0, \tau') \), that is, the \( s = 0 \) endpoint of the parabola has the same value for all tolls. Thus, for \( \tau \) satisfying (66), \( \mathcal{L}^\tau(G, S_L, \tau^*) < \mathcal{L}^\tau(G, S_U, \tau^*) \) if and only if \( \mathcal{L}^\tau(G, \frac{S_L + S_U}{2}, \tau^*) < \mathcal{L}^\tau(G, \frac{S_L + S_U}{2}, \tau^*) \).

By concavity, any tolls that result in globally optimal routing for \( s = \frac{S_L + S_U}{2} \) will also be optimal in the sense of (33). It is easily verified that for a known homogeneous sensitivity \( s \), any tolls \( \tau \) that satisfy

\[
H \left( \tau + b/(2s) \right) = 0
\]

result in globally optimal routing. The proof of this is obtained by substituting (67) into the gradient (with respect to \( \tau \)) of \( \mathcal{L}^\tau(G, s, \tau) \) and applying the facts from Observation 5.1.

Therefore, any \( \tau \) that satisfies (67) with \( s = \frac{S_L + S_U}{2} \) will be uncertainty optimal. That is, \( \tau^* \) satisfies (64). \( \square \)

Evaluating (57) with tolls satisfying (64) yields an expression for a Nash flow induced by \( \tau^* \) as a function of \( s \):

\[
f^\tau(G, s, \tau^*) = R + Hb \left( S_L + S_U - s \right) / (S_L + S_U)
\]

implying that \((R + Hb)\) specifies an untolled Nash flow. For parallel networks, it is easy to show that every element of \( R \) is nonnegative; thus, since \( \alpha \triangleq \frac{(S_L + S_U)}{N_0 + S_U} \in [0, 1] \), it must be that \((R + Hb\alpha)\) represents a feasible flow.

There are two possible worst-case flows using fixed toll \( \tau^* \):

- One when the sensitivity is \( S_L \), the other when the sensitivity is \( S_U \). In terms of (68), we write these flows as

\[
f^- = f^\tau(G, S_L, \tau^*) = R + Hb \left( S_U / (S_L + S_U) \right)
\]

\[
f^+ = f^\tau(G, S_U, \tau^*) = R + Hb \left( S_L / (S_L + S_U) \right)
\]

Next, we show that \( f^- \) and \( f^+ \), the worst-case flows for optimal fixed tolls, are actually exactly equal to worst-case flows achievable with scaled marginal-cost tolls (35) with a particular scalar. The machinery of Claim 5.1.1 describes the Nash flows \( f^{mc}(G, s, \kappa) \) resulting from homogeneous sensitivity \( s \) and marginal-cost tolls scaled by \( \kappa > 0 \):

\[
f^{mc}(G, s, \kappa) = R + Hb / (1 + s\kappa).
\]

The derivation of this is straightforward; it is detailed in [7].

The worst worst-case flows occur when the sensitivity of the population has been grossly over- or underestimated; for example, if a population with sensitivity \( S_L \) is using a network with \( \kappa = 1/S_L \) (and vice versa). There are two such flows:

\[
f^- = R + \frac{Hb}{1 + S_L / S_U} \quad \text{and} \quad f^+ = R + \frac{Hb}{1 + S_U / S_L}
\]

Comparing the above to (69) and (70), we see that \( f^- = f^- \) and \( f^+ = f^+ \). Thus, since

\[
f^\tau(G, S_L, \tau^*) = f^{mc}(G, S_L, 1/S_U)
\]

\[
f^\tau(G, S_U, \tau^*) = f^{mc}(G, S_U, 1/S_L)
\]
it must be true that (rewriting now in terms of affine tolls)
\[
L^\text{nf}(G, S_L, \tau^*) = L^\text{nf}(G, S_L, \tau^*(1/S_L, 0)) \tag{72}
\]
\[
L^\text{nf}(G, S_U, \tau^*) = L^\text{nf}(G, S_U, \tau^*(1/S_L, 0)). \tag{73}
\]
By design, (72) equals (73), so we have that
\[
\max_{s \in \{S_L, S_U\}} L^\text{nf}(G, s, \tau^*(1/S_L, 0)) = \max_{s \in \{S_L, S_U\}} L^\text{nf}(G, s, \tau^*).
\]

REFERENCES


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