Design Tradeoffs in Concave Cost-Sharing Games

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Abstract—This note focuses on the design of cost-sharing rules to optimize the efficiency of the resulting equilibria in cost-sharing games with concave cost functions. Our analysis focuses on two well-studied measures of efficiency, termed the price of anarchy and price of stability, which provide worst-case guarantees on the performance of the (worst or best) equilibria. Our first result characterizes the cost-sharing design that optimizes the price of anarchy, followed by the price of stability. This optimal cost-sharing rule is precisely the Shapley value cost-sharing rule. Our second result characterizes the cost-sharing design that optimizes the price of stability, followed by the price of anarchy. This optimal cost-sharing rule is precisely the marginal contribution cost-sharing rule. This analysis highlights a fundamental tradeoff between the price of anarchy and price of stability in the considered class of games. That is, given the optimality of both the Shapley value and marginal cost distribution rules in each of their respective domains, it is impossible to improve either the price of anarchy or price of stability without degrading its counterpart.

Index Terms—Distributed control, game theory, multiagent systems, networked control systems.

I. INTRODUCTION

It is well known that networked social systems can exhibit highly inefficient system behavior [16]. The root of these inefficiencies is twofold. First, these systems possess a decision-making architecture where the collective behavior emerges from actors making local independent decisions in response to personalized objective functions. Second, these actors may be responding to objective functions that are not well aligned with the system objective. A transportation network is one example of a social system that is problematic on both fronts, as traffic patterns emerge from individual drivers making routing choices aimed at minimizing their own experienced congestion (or some other personalized objective). The lack of alignment between the objectives of the system designer and drivers is a key contributor to the inefficiencies associated with the emergent traffic patterns.

The price of anarchy and price of stability are two well-studied performance metrics for studying the quality of equilibria in networked-social systems, or more generally, games [9]. Much of this literature has focused on analyzing these metrics for specific classes of games, where the game structure is chosen to model individual preferences in a system of interest; see recent monographs, book chapters, and references therein [8], [14], [16], [18]. More recently, research has focused on developing universal approaches for bounding these measures of efficiency [19], as well as characterizing classes of agent objective functions that meet certain efficiency standards [11], [21]. A commonality in most results pertaining to the study of equilibrium efficiency in games is the central focus on analysis. The justification for this analytical focus stems from the fact that a system-planner may have minimal means to design (or influence) the actors’ personalized objective functions.1

The field of networked control theory has necessitated looking at this literature from a new perspective. Like social systems, a networked control system (or multiagent system) has an underlying decision-making architecture where the agents make local independent decisions in response to local objective functions. Unlike social systems, the agents’ local objective functions represent one component of the design space geared at engineering desirable collective behavior. Accordingly, the central question from a controls perspective is how to design admissible agent objective functions so as to minimize the inefficiencies resulting from a networked decision-making architecture.

The complexity associated with evaluating the efficiency of equilibria has made characterizing the agent objective functions that optimize the efficiency of equilibria a daunting task. Nonetheless, optimal agent objective functions have been derived for very specific problem domains. One of the most noteworthy results is from [5] where the author derives local agent objective functions that minimize the price of anarchy for a class of set covering problems. Alternatively, [10] derives the set of local objective functions that minimize the price of anarchy for a class of network coding problems. Other results pertaining to the design of agent objectives have focused on ensuring equilibrium existence [11], [3], [7], [11], optimizing the price of anarchy when restricting attention to budget-balanced objective functions, [3], [4], [6], analyzing the efficiency guarantees associated with specific objective design methodologies [13], [17], as well as exploring taxation methodologies for augmenting agent objective functions in routing problems [2], [20]. It is important to highlight that all of the above results are strongly tied to the considered domain and do not necessarily generalize more broadly.

A classical result in this context comes from [18] where the authors characterize the price of anarchy for a class of routing problems where users are interested in minimizing their own experience congestion. For example, the work in [18] demonstrates that the price of anarchy is 4/3 in routing games where the edges have affine latency functions, i.e., a Nash equilibrium can have an aggregate congestions that is at most 33% higher than the aggregate congestion associated with the optimal allocation. The focus of this work is not on characterizing the efficiency guarantees associated with a given set of agent objective functions, but rather on identifying the agent objective functions that optimize these efficiency guarantees.
This note addresses the question of how to design local agent objective functions in cost-sharing games with concave cost functions. We begin by characterizing the local agent objective functions, which can be interpreted by a cost-sharing rule, that optimizes the price ofarchy, followed by the price of stability. Our characterization reveals that this optimal cost-sharing rule is precisely the Shapley value distribution rule that is extensively studied in the literature. The Shapley value distribution rule is budget balanced, meaning that the sum of the agents’ objective functions is equal to the global objective for any allocation. Many of the aforementioned results highlight the optimality of the Shapley value distribution rule when restricting attention to budget-balanced distribution rules, e.g., [3], [4], and [6]. Here, we demonstrate that the Shapley value distribution rule optimizes the price of anarchy over all rules, irrespective of whether or not a budget-balanced condition is specified. This result suggests that having budget-balanced distribution rules may be helpful for optimizing the price of anarchy.

Our second result characterizes the cost-sharing rule that optimizes the price of stability, followed by the price of anarchy. Our characterization reveals that this optimal cost-sharing rule is precisely the marginal contribution distribution rule that is also extensively studied in the literature [1], [3], [7], [11]. This analysis highlights a fundamental tradeoff between the price of anarchy and price of stability in the considered class of games. That is, given the optimality of both the Shapley value and marginal cost distribution rules, it is impossible to improve either the price of anarchy or price of stability without degrading its counterpart.

II. Model and Performance Metrics

In this note, we consider a class of networked resource allocation problems where the goal is to allocate a collection of agents \( N = \{1, \ldots, n\} \) to a finite set of resources \( R = \{r_1, r_2, \ldots, r_m\} \) to minimize a given system-level cost function \( C \). The set of possible assignments for each agent \( i \in N \) is given by \( A_i \subseteq 2^R \), and the global cost associated with an allocation \( a = (a_1, a_2, \ldots, a_n) \in A := A_1 \times \cdots \times A_n \) is of the form

\[
C(a) = \sum_{r \in R} v_r \cdot C_r([a]_r)
\]  

(1)

where \( C_r : \{0, 1, \ldots, n\} \rightarrow \mathbb{R} \) is a base cost function, \( v_r \geq 0 \) is the relative value associated with resource \( r \), and \([a]_r = \{j \in N : r \in a_j\}\) captures the number of agents assigned to resource \( r \) in the allocation \( a \). Furthermore, we focus on the class of base cost functions that are positive, increasing, and concave, i.e., for all \( k \geq 0 \) we have 1) \( C_r(k) \geq 0 \), 2) \( C_r(k + 1) \geq C_r(k) \), and 3) \( C_r(k + 1) - C_r(k) \geq C_r(k + 2) - C_r(k + 1) \). We will often express the cost at resource \( r \) as \( C_r([a]_r) \), as opposed to \( v_r \cdot C_r([a]_r) \), for notational convenience. Lastly, we adopt the convention that \( C_r(0) = 0 \) and \( C_r(1) = 1 \) without loss of generality.\(^2\)

This note focuses on the design of local agent objective functions that optimize the efficiency of the resulting equilibria. To that end, we consider the framework of cost-sharing games (or distributed welfare games [11]) where each agent is associated with a local cost function of the form

\[
J_i(a) = \sum_{r \in a_i} v_r \cdot f([a]_r)
\]

(2)

where \( f : \{0, 1, \ldots, n\} \rightarrow \mathbb{R} \), which we refer to as the base distribution rule, prescribes the cost to each agent for any allocation \( a \). Note that an agent’s cost function in (2) is inherently local as it only depends on the resources that the agent selected, the number of agents that also selected these resources, as well as the base distribution rule \( f \) and relevant resource values \( \{v_r\} \). We will express a resource allocation game as the tuple \( G = \{N, R, \{A_i\}_{i \in N}, f, C_r, \{v_r\}_{r \in R}\} \). We will often drop the subscripts on the above sets, e.g., denote \( \{v_r\}_{r \in R} \) as simply \( \{v_i\} \), for brevity. Lastly, we define the set of games induced by a given base distribution rule \( f \) and base cost function \( C_r \) as

\[
G_{f,C_r} = \{G = \{N, R, \{A_i\}_{i \in N}, f, C_r, \{v_r\}, v_i \geq 0\} : A_i \subseteq 2^R, v_i \geq 0\}.
\]

Note the set of induced games \( G_{f,C_r} \) allows for variations in the agents’ action sets, the resource valuations, as well as the number of agents and resources.\(^3\)

The goal of this note is to derive a base distribution rule \( f \) that yields desirable guarantees over the set of induced games \( G_{f,C_r} \). We focus on the equilibrium concepts of pure Nash equilibrium, which we will henceforth refer to as just an equilibrium, which is defined as an action profile \( a^* \in A \) that satisfies

\[
J_i(a^*_i, a^*_{-i}) = \min_{a_i \in A_i} J_i(a_i, a^*_{-i}), \quad \forall i \in N
\]

(3)

where \( a^*_{-i} = (a^*_1, \ldots, a^*_i, \ldots, a^*_n) \) captures the allocation choice of all agents not including agent \( i \). We will evaluate the quality of a base distribution rule by two worst-case measures of the efficiency of the equilibria in the induced games \( G_{f,C_r} \): price of anarchy and price of stability [9]. The price of anarchy associated with a base distribution rule \( f \) over the induced games \( G_{f,C_r} \) is defined as

\[
\text{PoA}(G_{f,C_r}) = \max_{G \in G_{f,C_r}} \min_{a \in A \cap \mathbb{R}^+} \left\{ \frac{C(a^{opt}; G)}{C(a; G)} \right\} \geq 1,
\]

(4)

where we use the notation \( a^{opt} \in G \) to mean an allocation \( a \in A \) that is an equilibrium in the game \( G \) and the notation \( (C(a;G), C(a^{opt}; G)) \) to denote the system-cost associated with the (equilibrium \( a^{opt} \), optimal allocation) in the game \( G \). The price of stability, on the other hand, provides a worst-case analysis over the best equilibrium in each game \( G \in G_{f,C_r} \), i.e.,

\[
\text{PoS}(G_{f,C_r}) = \max_{G \in G_{f,C_r}} \min_{a \in A \cap \mathbb{R}^+} \left\{ \frac{C(a^{opt}; G)}{C(a; G)} \right\} \geq 1.
\]

(5)

Note that \( \text{PoA}(G_{f,C_r}) \geq \text{PoS}(G_{f,C_r}) \geq 1 \).

III. Optimizing the Price of Anarchy

In this section, we focus on characterizing the distribution rule that optimizes the price of anarchy followed by the price of stability. To that end, let \( f^* = \arg \min_{f \in F^0} \text{PoA}(G_{f,C_r}) \) denote the family of distribution rules that achieves minimal price of anarchy over the set of induced games \( G_{f,C_r} \). Here, we focus on characterizing the distribution rule

\[
f^* = \arg \min_{f \in F^0} \{\text{PoS}(G_{f,C_r})\}.
\]

(6)

\(^2\)The restriction to a single structural form of the base cost function \( C_r \) is geared at understanding how the structure of the cost function impacts the resulting efficiency guarantees and the structure of the optimal design of agent objective functions. This model of resource allocation is consistent with the models studied in [3], [5], [6], and [11] among others. For example, [6] studies agent objective design for this exact model where \( C_r \) is convex instead of concave.

\(^3\)The set of games should be denoted by \( G_{f,C_r,n} \) to explicitly highlight the dependence on the number of agents \( n \). However, we express the set of games as merely \( G_{f,C_r} \) for brevity when the number of agents is clear.
The main result of this section demonstrates that the Shapley value distribution rule, which for any $k \geq 1$ is of the form
\[
 f^\text{sv}(k) = \frac{C_b(k)}{k}
\]
is the unique distribution rule that achieves these guarantees. This result is summarized in the following theorem.

**Theorem 3.1**: Let $C_b$ be any function that is positive, increasing, and concave. The Shapley value distribution rule $f^\text{sv}$ is the unique distribution rule that satisfies (6). The price of anarchy and price of stability guarantees associated with the Shapley value distribution rule are
\[
 \text{PoA}(G_{f^\text{sv}}, c_b) = \max_{m^* \leq m} \left\{ \sum_{k=1}^{m^*} \frac{C_b(k)}{k} \frac{m^*}{n} \right\},
\]
\[
 \text{PoS}(G_{f^\text{sv}}, c_b) = \frac{n}{C_b(n)}.
\]

### A. Preliminaries

We begin by presenting an approach for characterizing the price of anarchy, termed smoothness, introduced in [19] and mildly extended in [17]. Consider a family of games $G$ where for every game $G \in G$ and every action profile $a \in A$, $\sum_{i \in N} J_i(a) \leq q \cdot C(a; G)$ where $q > 0$. If there exists coefficients $\lambda > 0$ and $\mu < q$ such that for every game $G \in G$ and action profiles $a, a^* \in A$
\[
 \sum_{i \in N} J_i(a', a^{-i}) \leq \lambda \cdot C(a^*; G) + \mu \cdot C(a; G)
\]
then the resulting price of anarchy is bounded above by
\[
 \text{PoA}(G) \leq \frac{(\lambda/q)}{1 - (\mu/q)}.
\]
Hence, characterizing smoothness coefficients $(\lambda, \mu)$ that satisfies (10) provides an upper bound on the price of anarchy.

A second result we review focuses on bounding the price of stability in a class of games termed potential games [15]. A game $G$ is a potential game with potential function $\phi : A \to \mathbb{R}$ if for any allocation $a \in A$, agent $i \in N$, and alternative choice $a'_i \in A$, we have
\[
 J_i(a, a^{-i}) - J_i(a', a^{-i}) = \phi(a) - \phi(a').
\]
Suppose for any game $G \in G$, 1) the game is a potential game with potential function $\phi_G$, 2) there exists coefficients $q_1, q_2 > 0$ such that for every $a \in A$
\[
 q_1 \cdot C(a; G) \leq \phi_G(a) \leq q_2 \cdot C(a; G).
\]
Then by [16] (Chapter 17), the price of stability associated with the set of games $G$ satisfies
\[
 \text{PoS}(G) \leq \frac{q_2}{q_1}.
\]

### B. Proof of Theorem 3.1

We begin with an initial lemma that provides a lower bound on the best achievable price of anarchy by any anonymous base distribution rule.

**Lemma 3.1**: Let $G_{f, c_b}$ denote the family of games induced by any base distribution rule $f : \{0, 1, \ldots, n\} \to \mathbb{R}$ and base cost function $C_b$ that is positive, increasing, and concave. Then the price of anarchy satisfies
\[
 \text{PoA}(G_{f, c_b}) \geq \frac{n}{C_b(n)}.
\]

**Proof**: Consider an $n$ agent problem with $n+1$ resources where each resource $r_0, \ldots, r_{n+1}$ has a value 1, and each agent $i$ has an action set $A_i = \{\{r_i\}, \{r_{n+1}\}\}$. Irrespective of the base distribution rule $f$, the allocation where each agent $i \in N$ selects resource $\{r_i\}$ is an equilibrium, and the total cost of this equilibrium is $n$. The optimal allocation is when all agents select resource $\{r_{n+1}\}$, which yields a total cost of $C_b(n)$.

Lemma 3.1 provides a constraints on the minimum achievable price of anarchy for any anonymous base distribution rule $f$. Our second lemma highlights the implications of minimizing the price of anarchy on our second performance metric of interest, the price of stability.

**Lemma 3.2**: Let $F^\text{sv}$ be the set of anonymous distribution rules that provides a price of anarchy of exactly $n/C_b(n)$. For any $f \in F^\text{sv}$, the price of stability must satisfy
\[
 \text{PoS}(G_{f, c_b}) \geq \frac{n}{C_b(n)}.
\]

**Proof**: We begin by showing that it is necessary to have $f(k) > 0$ for all $k$ in order to achieve a finite price of anarchy. Suppose this was not the case, and $f(k) \leq 0$ for some $k \leq n$. Furthermore, let $k$ be the first entry where this is true. We will now construct an example that establishes the contradiction for $k = 1$, and the remaining cases $k > 1$ follow in the same fashion. To that end, consider an example with 1 agent and two resources $r_1$ and $r_2$ with valuations $v_1 = 1$ and $v_2 = 2 > 1$, where $A_1 = \{\{r_1\}, \{r_2\}\}$. The optimal allocation is when the agent selects resource $\{r_1\}$, which generates a total cost of 1. An equilibrium is when the agent selects resource $\{r_1\}$ which generates a total cost of $\alpha$. Hence, the efficiency of this equilibrium grows unbounded in $\alpha$, thereby showing that it is necessary to have $f(1) > 0$ to attain a finite price of anarchy.

We now move on to establishing a more stringent condition on the structure of the distribution rule $f$ that is necessary to satisfy a price of anarchy of $n/C_b(n)$. Note that any scaled version of $f$, i.e., a distribution rule $\alpha \cdot f$ where $\alpha > 0$, yields the exact same equilibria as $f$ since $f > 0$. Hence, we focus on distribution rules where $f(1) = 1$ without loss of generality. Consider a family of examples where for each $k \in \{1, \ldots, n\}$ we have $n - k + 2$ resources where each agent $i \in \{1, \ldots, n - k\}$ has an action set $A_i = \{\{r_i\}, \{r_{n-k+1}\}\}$, and each agent $i \in \{n - k + 1, \ldots, n\}$ has action set $A_i = \{\{r_{n-k+1}\}, \{r_{n-k+2}\}\}$, as illustrated in Fig. 1. Define the coefficient of resource $r_{n-k+1}$ as $1/f(k)$ and all other coefficients as 1. The allocation where $a_i = \{r_i\}$ for all $i \in \{1, \ldots, n - k\}$ and $a_i = \{r_{n-k+1}\}$ for all $i \in \{n - k + 1, \ldots, n\}$ is an equilibrium as each agent is receiving a cost of 1 at this allocation. The system cost associated with this allocation is $C_b(k)/f(k) + (n - k)$. The optimal allocation is when all agents select resource $r_{n-k+1}$ which yields a total system cost of $C_b(n)$. Since the price of anarchy is equal to $n/C_b(n)$, we know that
\[
 \frac{C_b(k)/f(k) + (n - k)}{C_b(n)} \leq \frac{n}{C_b(n)}.
\]
which gives us that
\[ f(k) \geq \frac{C_k(k)}{k}, \forall k. \]  
(17)

We now consider a second class of examples to identify an accompanying condition on the price of stability. For each \( z \leq n \), consider an example with \( z \) agents and \( z + 1 \) resources where each agent \( i \in \{1, \ldots, z\} \) is associated with an action set \( \mathcal{A}_i = \{\{r_i\}, \{r_{i+1}\}\} \). Furthermore, let the coefficient of resource \( r_{i+1} = 1 \) while the coefficients associated with each resource \( r_i \in \{r_1, \ldots, r_z\} \) be \( v_i = \min\{f(1), \ldots, f(i)\} - \epsilon \) where \( \epsilon > 0 \) is an arbitrarily small constant. The unique equilibrium associated with this example is when each agent \( i \in \{1, \ldots, z\} \) selects resource \( \{r_i\} \) which yields a total system cost of \( v_1 + \cdots + v_z \), where by (17) each \( v_i \) satisfies \( v_i \geq \min\{C_b(1)/1, \ldots, C_b(i)/i\} = C_b(i)/i \) due to the positivity and concavity of \( C_b \). An alternative, potentially optimal, allocation is when all users select resource \( r_{z+1} \), which yields a total system cost of \( C_b(z) \). Accordingly, the price of stability satisfies

\[ \text{PoS}(\mathcal{G}_{f,C_b}) \geq \max_{1 \leq k \leq n} \left\{ \frac{m^*}{k} \sum_{k=1}^{m^*} C_b \left( \frac{k}{C_b(m^*)} \right) \right\} \]
(18)

\[ \geq \max_{1 \leq k \leq n} \left\{ \frac{m^*}{k} \sum_{k=1}^{m^*} C_b \left( \frac{k}{C_b(m^*)} \right) \right\} \]
(19)

which completes the proof.

Lemma 3.2 demonstrates that any distribution rule that minimizes the price of anarchy inherits performance limitations with regards to the price of stability. Using these two lemma, we know return our attention to proving Theorem 3.1.

Proof of Theorem 3.1: Let \( G \in \mathcal{G}_{f,C_b} \) be any game induced by the Shapley value distribution rule. Turning our attention to the price of anarchy, we have by definition that \( \sum_i J_i(a) = C(a) \) for any agent \( a \in A \), hence \( q = 1 \) in the smoothness approach presented in Section III-A. With regards to the smoothness parameters, let \( a \) and \( a^* \) be any two action profiles. Evaluating the sum of the agents’ cost functions give us

\[ \sum_{i \in N} J_i(a^*_i, a_{-i}) = \sum_{i \in N} \sum_{r \in N} C_i \left( \frac{\left| a^*_i \right| + 1}{\left| a_{-i} \right| + 1} \right), \]

\[ \leq \sum_{i \in N} C_i(1), \]

\[ = \sum_{r \in N} \left( \frac{\left| a^*_i \right|}{C_i(\left| a^*_i \right|)} \right) C_i \left( \frac{\left| a^*_i \right|}{C_i(\left| a^*_i \right|)} \right), \]

\[ \leq \left( \frac{n}{C_b(n)} \right) C(a^*). \]

Hence, the price of anarchy satisfies \( \text{PoA}(\mathcal{G}_{f,C_b}) \leq n/C_b(n) \) and the tightness comes from Lemma 3.1.

Turning our attention to the price of stability, it is straightforward to show that the Shapley value distribution rule gives rise to a potential game with a potential function of the form

\[ \phi^{sv}(a) = \sum_{i \in N} \sum_{k=1}^{m^*} C_i \left( \frac{k}{C_b(m^*)} \right) \]
\[ = \sum_{r \in R} C_i \left( \frac{\left| a^*_i \right|}{k} \right) \sum_{k=1}^{m^*} \left( \frac{C_b \left( \frac{k}{C_b(m^*)} \right)}{k} \right). \]

\[ \leq C(a) \cdot \max_{1 \leq k \leq n} \left\{ \frac{m^*}{k} \sum_{k=1}^{m^*} \left( \frac{C_b \left( \frac{k}{C_b(m^*)} \right)}{k} \right) \right\}. \]
(21)

Consequently, an upper bound associated with the price of stability of the Shapley value distribution rule from (14) is

\[ \text{PoS}(\mathcal{G}_{f,C_b}) \leq \max_{1 \leq k \leq n} \left\{ \frac{m^*}{k} \sum_{k=1}^{m^*} \left( \frac{C_b \left( \frac{k}{C_b(m^*)} \right)}{k} \right) \right\}. \]

The tightness of the price of stability comes from Lemma 3.2. Lastly, the uniqueness of this distribution rule is due to (17), as the inequality in (19) in the proof of Lemma 3.2 holds if and only if \( f(k) = C_b(k)/k \).

IV. OPTIMIZING THE PRICE OF STABILITY

In this section, we focus on characterizing the distribution rule that optimizes the price of stability followed by the price of anarchy. To that end, let \( F^* = \arg \min_{f \in F} \text{PoS}(\mathcal{G}_{f,C_b}) \) denote the family of distribution rules that achieves minimal price of stability over the set of induced games \( \mathcal{G}_{f,C_b} \). Here, we focus on characterizing the distribution rule that satisfies

\[ f^* = \arg \min_{f \in F^*} \text{PoA}(\mathcal{G}_{f,C_b}). \]
(22)

The main result of this section demonstrates that the marginal contribution distribution rule, which for any \( k \geq 1 \) is of the form

\[ f^\text{mc}(k) = C_b(k) - C_b(k-1) \]
(23)

is the unique distribution rule that achieves these guarantees. This result is summarized in the following theorem.

Theorem 4.1: Let \( C_b \) be any function that is positive, increasing, and concave. The marginal contribution distribution rule \( f^\text{mc} \) is the unique distribution rule that satisfies (22). The price of anarchy and price of stability guarantees associated with the marginal contribution distribution satisfy

\[ \text{PoS}(\mathcal{G}_{f^\text{mc},C_b}) = 1, \]
(24)

\[ \text{PoA}(\mathcal{G}_{f^\text{mc},C_b}) \geq \frac{1}{C_b(n) - C_b(n-1)}. \]
(25)

\[ \text{PoA}(\mathcal{G}_{f^\text{mc},C_b}) \leq \max_{1 \leq k \leq n} \left\{ \frac{(n/C_b(n)) \cdot C_b(k)}{k \cdot (C_b(k) - C_b(k-1))} \right\}. \]
(26)

Proof: It is well known that the marginal contribution distribution rule guarantees a price of stability of \( 1 \), i.e., \( \text{PoS}(\mathcal{G}_{f^\text{mc},C_b}) = 1 \) [11]. We begin by demonstrating some necessary conditions on the structure of \( f \) to ensure that \( \text{PoS}(\mathcal{G}_{f,C_b}) = 1 \). First suppose \( f(k) > C_b(k) - C_b(k-1) \) for some \( k \leq n \). Consider a game \( G \in \mathcal{G}_{f,C_b} \) with \( k \) agents and two resources \( \{r_1, r_2\} \) with valuations \( v_{r_1} = 1 \) and \( v_{r_2} \) that satisfies \( f(k) > v_{r_1} > C_b(k) - C_b(k-1) \). Further, suppose each agent \( i \in \{1, \ldots, k-1\} \) has action set \( \mathcal{A}_i = \{r_1\} \) and agent \( k \) has action set \( \mathcal{A}_k = \{r_2\} \). Note that the unique equilibrium for this game is when all agents select \( r_1 \) and the optimal allocation is when agent \( k \) selects \( r_2 \). Hence, the \( \text{PoS}(\mathcal{G}_{f,C_b}) \geq 1 \) contradicting our assumption. A similar example can be derived to show that if \( f(k) < C_b(k) - C_b(k-1) \), then the \( \text{PoS}(\mathcal{G}_{f,C_b}) > 1 \). Hence, \( f^\text{mc} \) is the unique distribution rule that achieves \( \text{PoS}(\mathcal{G}_{f^\text{mc},C_b}) = 1 \).

We now turn our attention to characterizing the price of anarchy for the marginal contribution distribution rule \( f^\text{mc} \). To that end, consider
any game $G \in \mathcal{G}_{n,c,C_b}$ and let $a$ and $a^*$ be any two action profiles. Focusing on the sum of the agents’ cost functions reveals

$$\sum_{i \in N} J_i(a_i^*, a_{-i}) \leq \sum_{i \in N} \sum_{a_i} C_i(1),$$

$$= \sum_{r \in R} C_r(|a^*_r|) \left( \frac{|a^*_r| \cdot C_r(1)}{C_r(|a^*_r|)} \right),$$

$$\leq \left( \frac{n}{C_b(n)} \right) C(a^*)$$

where the first step follows from the concavity of $C_r$. Hence, the game is smooth with parameters $\lambda = n/C_b(n)$ and $\mu = 0$.

Turning attention to characterizing the term $q$ in (11), for any $a \in A$ we have

$$\sum_{i \in N} J_i(a) = \sum_{i \in N} \sum_{r \in A_i} C_i(|a_r|) - C_i(|a_r| - 1),$$

$$= \sum_{r \in R} |a_r| \cdot (C_r(|a_r|) - C_r(|a_r| - 1)), $$

$$= \sum_{r \in R} C_r(|a_r|) \cdot \left( \frac{|a_r| - |a_r| \cdot C_r(|a_r| - 1)}{C_r(|a_r|)} \right),$$

$$\geq \min_{1 \leq k \leq n} \left\{ k - \frac{k \cdot C_b(k - 1)}{C_b(k)} \right\} C(a).$$

Letting $q = \min_{1 \leq k \leq n} \left\{ k - \frac{k \cdot C_b(k - 1)}{C_b(k)} \right\}$ and using (11) we have

$$\text{PoA}(G_{n,c,C_b}) \leq \max_{1 \leq k \leq n} \left\{ \left( \frac{n}{C_b(n)} \right) \cdot \left( \frac{C_b(k)}{k \cdot (C_b(k) - C_b(k - 1))} \right) \right\}. \quad (27)$$

To establish the lower bound on the price of anarchy given in (25), we will now construct a game $G \in \mathcal{G}_{n,c,C_b}$ with a price of anarchy equal to this bound. To that end, consider an $n$ player game with two resources $R = \{r_1, r_2\}$ where $v_{r_1} = 1$, $v_{r_2} = (C_b(n) - C_b(n - 1))$, and the action sets satisfy $A_i = \{\{r_1\}, \{r_2\}\}$ for all $i \in N$. Consider the allocation where all players choose $r_1$, i.e., $a = (r_1, \ldots, r_1)$. This allocation is an equilibrium as no player has a unilateral incentive to deviate, and the cost associated with this allocation is $C(a) = C_b(n)$. The optimal allocation is when all players choose $r_2$ which yields a total system cost of $(C_b(n) - C_b(n - 1))C_b(n)$. Hence, the efficiency of this equilibrium is $1/(C_b(n) - C_b(n - 1)).$

## V. Simulations

In this section, we consider a concrete example to shed light on the developments above. Here, we focus on a base cost function $C_b(k) = k^d$ where $d \in [0, 1]$. Following Theorem 3.1, the efficiency guarantees associated with the Shapley value distribution rule are

$$\text{PoS}(G_{n,c,C_b}) = \max_{m \leq n} \left\{ \sum_{k=1}^m \frac{k^{d-1}}{m^d} \right\},$$

$$\text{PoA}(G_{n,c,C_b}) = n^{1-d}. $$

Following Theorem 4.1, the efficiency guarantees associated with the marginal contribution distribution rule are $\text{PoS}(G_{n,c,C_b}) = 1$ and

$$\max_{k \leq n} \left\{ \frac{(n/k)^{1-d}}{k^d - (k - 1)^d} \right\} \geq \text{PoA}(G_{n,c,C_b}) \geq \frac{1}{n^d - (n - 1)^d}.$$

For this particular example, it turns out that the lower bound and upper bound for $\text{PoA}(G_{n,c,C_b})$ are equivalent, but in general this need not be the case. Fig. 2 highlights the price of anarchy and price of stability guarantees for both the marginal contribution and Shapley value distribution rules for the considered class of problems with $n = 10$ agents and $d \in [0, 1]$. Note that there is an inherent tradeoff between the price of anarchy and price of stability. A byproduct of using a rule with minimal price of stability, i.e., the marginal contribution distribution rule, is a relatively poor price of anarchy. On the other hand, a byproduct of using a rule with minimal price of anarchy, i.e., the Shapley value distribution rule, is a relatively poor price of stability. Note that given the optimality of both the Shapley value and marginal cost distribution rules, it is impossible to improve either the price of anarchy or price of stability without degrading its counterpart.

## VI. Conclusion

This note focuses on the design of local agent cost functions in cost-sharing games. There are very few results in the literature that identify the agent objective functions that optimize the price of anarchy. In fact, most the results in the existing literature on optimal designs for cost-sharing games impose constraints on the underlying cost-sharing rules, e.g., budget balanced, which simplify the underlying analysis. In our setting where there was no budget-balanced constraints, we proved that the cost-sharing rule that optimizes the price of anarchy (followed by the price of stability) was in fact budget balanced. Whether or not having budget balanced rules is essential for attaining desirable price of anarchy guarantees in broader classes of problems warrants future research attention.

A second takeaway from this work is the potential tradeoff between the price of anarchy and price of stability in cost-sharing designs. In cost-sharing games with concave cost functions, there was indeed a tradeoff as evident by the optimality of the Shapley value and marginal contribution distribution rules in each of their respective domains. This analysis revealed that it is impossible to improve either the price of anarchy or price of stability guarantees of these optimal rules without degrading its counterpart. Alternatively, the marginal contribution and Shapley value distribution rules provide optimal “bookend” guarantees with regards to the price of anarchy and price of stability guarantees. More formally understanding the tradeoff between the price of anarchy and price of stability in cost-sharing designs will be a focus of future work.
REFERENCES


