Abstract—Networked systems are ubiquitous in today’s world with examples spanning from ecology to the social and engineering sciences. Much of the research in networked systems is analytical, where the focus is on characterizing (and potentially influencing) the emergent collective behavior. A more recent trend of research focuses on the design of networked systems capable of achieving diverse and highly coordinated collective behavior in the absence of centralized control. Focusing on the well-studied class of maximum coverage problems, our first result demonstrates that any agent-based algorithm relying solely on local information induces a fundamental trade-off between the best and worst case performance guarantees, as measured by the price of anarchy and price of stability. Our second result demonstrates how to use an additional piece of system-level information to breach these limitations, thereby improving the system’s performance.

I. INTRODUCTION

A multiagent system can be characterized by a collection of individual subsystems, each making independent decisions in response to locally available information. Such a decision-making architecture can either emerge naturally as the results of self-interested behavior, e.g., drivers in a transportation network, or be the result of a design choice in engineered system. In the latter case, the need for distributed decision-making stems from the scale, spatial distribution, and sheer quantity of information associated with various problem domains that exclude the possibility for centralized decision making and control. One concrete example is the problem of monitoring the perimeter of a wild fire, where the goal is to deploy a collection of unmanned aerial vehicles (UAV) to effectively survey the perimeter of a wild fire under the operational constraint that each UAV makes independent surveillance decisions in response to local information regarding its own aerial view of the landscape and minimal information regarding the state of neighboring UAVs [1]. Alternative examples include the use of robotic networks in post-disaster environments [15], [20], task scheduling and management [8], water conservative food production [18], fleets of autonomous vehicles [36], and micro-scale medical treatments [14], [35].

Regardless of the specific problem domain, the central goal in the design of a networked control system is to derive admissible control policies for the decision-making entities that ensure the emergent collective behavior is desirable with regards to a given system-level objective. At a high level, this design process entails specifying two key elements: the information available to each subsystem, attained either through sensing or communication, and a decision-making mechanism that prescribes how each subsystem processes available information to take decisions. The quality of a networked control architecture is ultimately gauged by several dimensions including the stability and efficacy of the emergent collective behavior, characteristics of the transient behavior, in addition to communication costs associated with propagating information throughout the system. The focus of this paper is on the following two questions associated with the design of networked control systems.

(i) What are the decision-making rules that optimize the performance of the emergent collective behavior for a given level of informational availability?

(ii) What is the value of information in networked control architectures? That is, how does informational availability translate to attainable performance guarantees for the emergent behavior through the design of appropriate decision-making mechanisms?

This paper seeks to shed light on the answer to these two questions in a class of multiagent maximum coverage problems introduced in [12]. In a multiagent maximum covering problem we are given a ground set of resources, and \( n \) collections of subsets of the ground set. Every resource is associated with a respective value or worth. The system-level objective is to select one set from each collection so as to maximize the total value of covered elements. It is important to highlight that there are well-established centralized algorithms that can derive an admissible allocation of agents to resources that is within a factor of \( 1 - 1/e \) of the optimal allocation’s value in polynomial time, provided that the \( P = N/P \). Unfortunately, the applicability of such centralized algorithms for the control of multiagent systems is limited given the concerns highlighted above.

This paper focuses on distributed approaches for reaching a near-optimal allocation where the individual agents make their covering selections in response to locally available information accordingly to a designed decision-making policy. The central goal here is to design agent decision-making rules that optimize the quality of the emergent collective behavior for a given level of informational availability. Of specific interest will be identifying how the level of information available to the individual agents impacts the attainable performance guarantees associated with the corresponding optimal networked control system.

In the spirit of [12], [26], we approach this problem through a game theoretic lens where we model the individual agents

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as players in a game and each agent is associated with a local objective function that guides its decision-making process. We treat these local objective functions as our design parameter and focus our analysis on characterizing the performance guarantees associated with the resulting equilibria of the designed game. Here, we model the informational restrictions discussed above as limitations on the amount of information that these local agent objective functions can depend on. We concentrate our analysis on two well-studied performance metrics in the game theoretic literature termed the price of anarchy and price of stability [2], [17]. Informally, the price of anarchy provides performance guarantees associated with the worst performing equilibrium relative to the optimal allocation. The price of stability, on the other hand, provides similar performance guarantees when restricting attention to the best performing equilibrium. The lack of uniqueness of equilibria implies that these bounds are often quite different.1

The work of [12] was one of the first to view price of anarchy as a design objective rather than its more traditional analytical counterpart. The main results in [12] identify a set of agent objective functions that optimize the price of anarchy when agents are only aware of (i) the resources the agent can select and (ii) the number of agents covering these resources. Note that in this setting, any agent i is unaware of the covering options of any other agents j ≠ i, as well as any resource values that the agent is unable to cover. Interestingly, [12] demonstrates that this optimal price of anarchy attains the same 1 − 1/e guarantees of the best centralized algorithms (even when the n collections of subsets are different), meaning that there is no degradation in terms of the worst-case efficiency guarantees when transitioning from the best centralized algorithm to the presented distributed algorithm that adheres to the prescribed informational limitations.

Our Contribution. The first question we address is whether utilizing the agent objective functions that optimize the price of anarchy has any unintended consequences with regards to other performance metrics of interest. The response is in the affirmative, as detailed in our first main theorem.

− In Theorem 3.1, we demonstrate that there is a fundamental trade-off between the price of anarchy and price of stability in such multiagent covering problems. That is, designing agent objective functions to improve the worst-case performance guarantees necessarily degrades the best-case performance guarantees. As corner cases, we demonstrate that any objective functions that ensure a price of anarchy of 1 − 1/e also inherit a price of stability of 1 − 1/e. Note that having a price of stability smaller than 1 implies that the optimal allocation is not necessarily an equilibrium. Alternatively, any objective functions that ensure a price of stability of 1 also inherit a price of anarchy of at most 1/2. This theorem characterizes the price of anarchy and price of stability frontier that is achievable through the design of agent objective functions in these multiagent covering problem.

The second main result of this manuscript focuses on the impact of information on the performance guarantees associated with the corresponding optimal agent objective functions. Theorem 3.1 demonstrates that there is a price of anarchy and price of stability frontier when agents are only aware of (i) the resources the agent can select and (ii) the number of agents covering these resources. The following theorem demonstrates that one can move beyond this frontier by providing the agents with additional information about the system at large.

− In Theorem 4.1, we identify a minimal (and easily attainable) piece of system-level information that can permit the realization of decision-making rules with performance guarantees beyond the price of anarchy / price of stability frontier provided in Theorem 3.1. When agents are provided with this additional information, which can be vaguely interpreted as the largest value of an uncovered resource in the system, one can derive agent objective functions that yield a price of anarchy of 1 − 1/e and a price of stability of 1, which was unattainable without this additional piece of information.

The importance of this result centers on the fact that certain attributes of the system can be exploited in networked control architectures if that information is propagated to the agents. Hence, there are notable performance gains associated with propagating that piece of information throughout the system. In other words, this piece of system-level information has value with regards to the task of multiagent coordination. Minimizing the amount of information that needs to be propagated throughout the system to move beyond this frontier is clearly an important question that warrants future attention.

Related Work. The results contained in this manuscript add to the growing literature of utility design, which can be interpreted as a subfield of mechanism design [7] where the objective is to design admissible agent objective functions to optimize various performance metrics, such as the price of anarchy and price of stability, [5], [16], [21], [27]. While recent work in [13] has identified all design approaches that ensure equilibrium existence in local utility designs, in general the question of optimizing the worst-case efficiency of the resulting equilibria, i.e., optimizing the price of anarchy, is far less understood. Nonetheless, there are a few positive results in this domain worth reviewing. Beyond [12], alternative problem domains where optimizing the price of anarchy has been explored include concave cost sharing games [24], and reverse carpooling games [23]. More recently, the authors in [32], [33] characterize and optimize the price of anarchy relative to a broader class of submodular and supermodular combinatorial optimization problems, rediscovering [12] as a special case. Lastly, a recent result in [10] characterizes a similar trade-off between the price of anarchy and price of stability in a mechanism design setting.

It is important to highlight that the bulk of the research regarding optimal utility design has concentrated on a specific class of objectives, termed budget-balanced objectives, which imposes the constraint that the sum of the agents’ objectives is equal to the system welfare for every allocation. Within the confines of budget-balanced agent objective functions, several works have identified the optimality of the Shapley value objective design with regards to the price of anarchy guarantees [6], [37], [38]. However, the imposition of the budget-balanced constraint is unwarranted in the context of

1The justification for focusing purely on equilibria, as opposed to dynamics, derives from the fact that there is a rich body of literature in distributed learning that could be coupled with the derived objective functions to attain distributed algorithms that lead the collective behavior to an equilibrium, c.f., [11].
multiagent system design and its removal allows for improved performance, as shown in [12] and this manuscript.

**Notation.** We use \( \mathbb{N}, \mathbb{R}_{>0} \) and \( \mathbb{R}_{\geq 0} \) to denote the set of natural, positive and non negative real numbers; \( e \) is Euler’s number.

## II. Model and Performance Metrics

In this section we introduce the multiagent maximum coverage problem and our game theoretic model for the design of local decision-making mechanisms [12]. Further, we define the objectives and performance metrics of interest, as well as provide a review of the relevant literature.

### A. Covering problems

Let \( \mathcal{R} = \{r_1, r_2, \ldots, r_m\} \) be a finite set of resources where each resource \( r \in \mathcal{R} \) is associated with a value \( v_r \geq 0 \) defining its importance. We consider a covering problem where the goal is to allocate a collection of agents \( N = \{1, \ldots, n\} \) to resources in \( \mathcal{R} \) in order to maximize the cumulative value of the covered resources. The set of possible assignments for each agent \( i \in N \) is given by \( A_i \subseteq \mathcal{R} \) and we express an assignment by the tuple \( a = (a_1, a_2, \ldots, a_n) \in A = A_1 \times \cdots \times A_n \). The total value, or welfare, associated with an allocation \( a \) is given by

\[
W(a) = \sum_{r \in \cup_{i \in N} a_i} v_r. \tag{1}
\]

The goal of the covering problem is to find an optimal allocation, i.e., an allocation \( a^{opt} \in A \) such that \( W(a^{opt}) \geq W(a) \) for all \( a \in A \). We will express an allocation \( a \) as \( (a_i, a_{\sim i}) \) with the understanding that \( a_{\sim i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \) denotes the collection of choices of the agents other than agent \( i \).

### B. A game theoretic model

This paper focuses on deriving distributed mechanisms for attaining near optimal solutions to covering problem where the individual agents make independent choices in response to local available information. Specifically, in this section we assume that each agent \( i \) has information only regarding the resources that the agent can select. Rather than directly specifying a decision-making process, here we focus on the design of local agent objective functions that adhere to these informational dependencies and will ultimately be used to guide the agents’ selection process. To that end, we consider the framework proposed in [12] where each agent is associated with a local utility or objective function \( U_i : \mathcal{A} \rightarrow \mathbb{R} \), and for any allocation \( a = (a_i, a_{\sim i}) \in A \), the utility of agent \( i \) is

\[
U_i(a_i, a_{\sim i}) = \sum_{r \in a_i} v_r \cdot f(|a|r), \tag{2}
\]

where \( |a|r \) captures the number of agents that choose resource \( r \) in the allocation \( a \), i.e., the cardinality of the set \( \{i \in N : a_i = r\} \), and \( f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \) defines the fractional benefit awarded to each agent for selecting a given resource in allocation \( a \). We will refer to \( f \) as the distribution rule throughout. Note that an agent’s utility function in (2) is consistent with the local information available as it only depends on the resource that the agent selected, the number of agents that also selected this resource, as well as the distribution rule \( f \) and relevant resource value \( v_r \). We will express such a \( n \)-agent welfare sharing game by the tuple \( G^n = (N, \mathcal{R}, \{A_i\}_{i \in N}, f, \{v_r\}_{r \in \mathcal{R}}) \) and drop the subscripts on the above sets, e.g., denote \( \{v_r\}_{r \in \mathcal{R}} \) as simply \( \{v_r\} \), for brevity.

The goal of this paper is to derive the distribution rule \( f \) that optimizes the performance of the emergent collective behavior. Here, we focus on the concept of pure Nash equilibrium as a model for this emergent collective behavior [29]. A pure Nash equilibrium, which we will henceforth refer to as just an equilibrium, is defined as an allocation \( a_{ne} \in A \) such that for all \( i \in N \) and for all \( a_i \in A_i \), we have

\[
U_i(a_i, a_{ne}) \geq U_i(a_i, a_{ne}^i). \tag{3}
\]

In essence, an equilibrium represents an allocation for which no single agent has a unilateral incentive to alter its covering choice given the choices of the other agents. It is important to highlight that an equilibrium might not exist in a general game \( G^n \). Nevertheless, when restricting attention to the class of games with utility functions defined in (2), an equilibrium is guaranteed to exist as the resulting game is known to be a congestion game [28].

We will measure the efficiency of an equilibrium allocation in a game \( G^n \) through two commonly studied measures, termed price of anarchy and price of stability, defined as follows:

\[
\text{PoA}(G^n) = \min_{a^{ne} \in G^n} \frac{W(a^{ne})}{W(a^{opt})} \leq 1, \tag{4}
\]

\[
\text{PoS}(G^n) = \max_{a^{ne} \in G^n} \frac{W(a^{ne})}{W(a^{opt})} \leq 1, \tag{5}
\]

where we use the notation \( a_{ne} \in G^n \) to imply an equilibrium of the game \( G^n \). In words, the price of anarchy characterizes the performance of the worst equilibrium of \( G^n \) relative to the performance of the optimal allocation, while the price of stability focuses on the best equilibrium in the game \( G^n \). Such distinction is required as equilibria are guaranteed to exists for the class of utilities considered in (2), but in general they are not unique. By definition \( 0 \leq \text{PoA}(G^n) \leq \text{PoS}(G^n) \leq 1 \).

Throughout, we require that a system designer commits to a distribution rule without explicit knowledge of the agent set \( N \), resource set \( \mathcal{R} \), action sets \( \{A_i\} \), and resource valuations \( \{v_r\} \). Note that once a particular distribution rule \( f \) has been chosen, this distribution rule defines a game for any realization of the parameters. The objective of the system designer is to provide desirable performance guarantees irrespective of the realization of these parameter, even if they were chosen by an adversary. To that end, let \( G^n_p \) denote the family of \( n \)-agent games induced by a given distribution rule \( f \), i.e., any game...
\(G^n \in G^n_f\) is of the above form. We will measure the quality of a distribution rule \(f\) by a worst-case analysis over the set of induced games \(G^n_f\), which is the natural extension of the price of anarchy and price of stability defined above, i.e.,
\[
\text{PoA}(G^n_f) = \min_{G^n \in G^n_f} \text{PoA}(G^n),
\]
\[
\text{PoS}(G^n_f) = \min_{G^n \in G^n_f} \text{PoS}(G^n).
\]

The price of anarchy \(\text{PoA}(G^n_f)\) for a given distribution rule \(f\) provides a bound on the quality of any equilibrium irrespective of the agent set \(N\), resource set \(\mathcal{R}\), action sets \(\{A_i\}\), and resource valuations \(\{v_i\}\). The price of stability, on the other hand, provides similar performance guarantees when restricting attention to the best equilibrium.\(^4\)

### C. An illustrative example

In this section we consider a simple family of single-selection covering problems consisting of two agents, \(N = \{1, 2\}\), and three resources, \(\mathcal{R} = \{r_1, r_2, r_3\}\), to illustrate the challenges that a system designer faces when designing a distribution rule \(f\) without explicit knowledge of the action sets \(A_1\) and \(A_2\) and the resource valuations \(v_1, v_2,\) and \(v_3\).

We begin by investigating the efficiency guarantees associated with the unique anonymous budget-balanced distribution rule, termed the equal share distribution rule, defined as
\[
f^{es}(j) = \frac{1}{j}, \quad \forall j \geq 1.
\]

Consider one problem instance where \(A_1 = \{\{r_1\}\}, A_2 = \{\{r_1\}, \{r_2\}\}\), and \(v_1 = 2, v_2 = 0.99,\) and \(v_3 = 0.\) Note that agent 1 is fixed at resource \(r_1\) for this instance. The payoff matrix and system welfare are given in Figure 1. In each box of the payoff matrix, the first entry is the payoff to agent 1 and the second entry is the payoff to agent 2. Note that for this instance, there is a unique equilibrium \((r_1, r_1)\) which garners a total welfare of 2. The optimal total welfare is 2.99; hence the price of anarchy and price of stability satisfy \(\text{PoA}(G^n_{f^{es}}) \leq 2/2.99 \approx 0.67\) and \(\text{PoS}(G^n_{f^{es}}) \leq 0.67.\)

Consider an alternative distribution rule, termed marginal contribution, that is of the form
\[
f^{mc}(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Consider an alternative problem instance where \(A_1 = \{\{r_1\}, \{r_2\}\}, A_2 = \{\{r_2\}, \{r_3\}\}\), and \(v_1 = 1.99, v_2 = 2,\) and \(v_3 = 0.01.\) The payoff matrix and system welfare are also given in Figure 2. Note that for this instance, there are now two equilibria, namely \((r_1, r_2)\) and \((r_2, r_3)\), while the optimal allocation is \((r_1, r_2).\) Hence the price of anarchy and price of stability must satisfy \(\text{PoA}(G^n_{f^{mc}}) \leq 2.01/3.99 \approx 0.5\) and \(\text{PoS}(G^n_{f^{mc}}) \leq 1.\)

These case studies suggest that \(f^{mc}\) outperforms \(f^{es}\) with regards to the price of stability, while \(f^{es}\) outperforms \(f^{mc}\) with regards to the price of anarchy. However, whether or not these conclusion hold more generally for alternative instances is unclear. While specific examples provide upper bounds on the price of stability / price of anarchy, a completely different set of techniques will have to be employed to provide lower bounds on these efficiency measures. Characterizing both the lower and upper bounds associated with these efficiency measures is essential to identify the optimal distribution rules in (3) and (4).

Lastly, a natural question is whether the locality requirement specified in the agents' utility functions in (2) is detrimental from a performance perspective. To address this question, suppose that a system designer was able to directly set each agent’s utility function as the global welfare, i.e., \(U_i = W\), which clearly violates our locality condition. Observe that, within the last example highlighted above, \(U_i = W\) produces the same equilibria that \(f^{mc}\) induces. Hence, the price of anarchy is at best 0.5 for the case when \(U_i = W\). As we will see in the ensuing section, there are alternative utility designs conforming to (2) that guarantee a better price of anarchy than 0.5. Hence, while setting \(U_i = W\) might seem a natural choice to optimize the price of anarchy, there are alternative choices that are local and yield far better results.

### III. THE TRADE-OFF BETWEEN THE PRICE OF STABILITY AND PRICE OF ANARCHY

In this section we provide our first main result that characterizes the inherent tension between the price of anarchy and price of stability as design objectives in multiagent maximum coverage problems. Here, we use the notation \(G_f = \cup_{n \geq 1} G^n_f\) to identify all maximum coverage games with an arbitrary number of players. Clearly, the price of anarchy (and similarly the price of stability) satisfies \(\text{PoA}(G_f) \leq \text{PoA}(G^n_f)\) for any \(n.\)

**Theorem 3.1:** Consider the class of maximum coverage games introduced in Section II. The following conclusions hold:

(i) The optimal price of anarchy satisfies
\[
\max_f \text{PoA}(G_f) = 1 - 1/e.
\]

(ii) Given a desired price of anarchy \(\alpha \in [0, 1/2]\), the best

Note that \(G^n_f\) is well-defined for any \(n\) since the distribution rule \(f\) is of the form \(f : N \rightarrow \mathbb{R}.\).
attainable price of stability satisfies
\[ \max_{f : \text{PoA}(G_f) \geq \alpha} \text{PoS}(G_f) = 1. \] (8)

(iii) Given a desired price of anarchy \( \alpha \in (1/2, 1 - 1/e) \) and \( n \geq 2 \), the best attainable price of stability satisfies
\[ \max_{f : \text{PoA}(G_f) \geq \alpha} \text{PoS}(G_f^\alpha) \leq Z(\alpha, n), \] (9)
where \( Z(\alpha, n) \) equals
\[ \frac{1}{1 + \max_{1 \leq j \leq n-1} j! \left( 1 - \left( \frac{1}{\alpha} - 1 \right) \sum_{i=1}^{j} \frac{1}{i!} \right) }. \] (10)

(iv) Given a desired price of anarchy \( \alpha = 1 - 1/e \), the best attainable price of stability satisfies
\[ \max_{f : \text{PoA}(G_f) \geq 1 - 1/e} \text{PoS}(G_f) = 1 - 1/e. \] (11)

(v) The bound in (9) is satisfied with equality if we restrict attention to single-selection maximum coverage games where \( A_i \subseteq R \), as opposed to \( A_i \subseteq 2^R \), for each agent \( i \in N \).

The results of Theorem 3.1 are illustrated in Figure 3. In particular, Theorem 3.1 establishes that there does not exist a distribution rule \( f \) that attains a price of stability and price of anarchy in the red region of the figure. Hence, there is an inherent tension between these two measures of efficiency as improving the performance of the worst equilibria necessarily comes at the expense of the best equilibria, and vice versa. The expression of \( Z(\alpha, n) \) defines this trade-off and satisfies \( \lim_{n \to \infty} Z(1 - 1/e, n) = 1 - 1/e \). The last result demonstrates that this trade-off curve is tight in the context of single-selection maximum coverage games, i.e., there are distribution rules \( f \) that achieve joint price of anarchy and price of stability guarantees highlighted on the curve \( Z(\alpha, n) \).

While the proof of Theorem 3.1 is presented in the Appendix, here we highlight a few conclusions from the proof that are worthy of discussion. First, the unique distribution rule that optimizes the price of stability, i.e., achieves \( \text{PoS}(G_f^\alpha) = 1 \), is the marginal contribution rule in (6). However, the price of anarchy satisfies \( \text{PoA}(G_f^\alpha) = 1/2 \). Second, the unique distribution rule that optimizes the price of anarchy for a given \( n \) is of the form
\[ f^*(j) = (j - 1)! \left( \frac{1}{(n-1)(n-1)!} + \sum_{i=1}^{n-2} \frac{1}{i!} \right), \quad j \leq n, \] (12)
and the accompanying price of anarchy guarantees are
\[ \text{PoA} \left( G_f^\alpha \right) = 1 - \frac{1}{(n-1)(n-1)!} + \sum_{i=0}^{n-1} \frac{1}{i!}. \] (13)

Further, letting \( n \to \infty \) gives us \( \text{PoA}(G_{f^*}) = 1 - 1/e \). However, Theorem 3.1 demonstrates that an unintended consequence associated with using \( f^* \) is that the price of stability also satisfies \( \text{PoS}(G_{f^*}) = 1 - 1/e \). Lastly, one could fine tune the above analysis for cardinality restricted games, as in [12], where there are limits on maximum number of agents that can select a given resource, i.e., \( \max_{r \in R, a \in A} |a_r| \leq k < n \), to obtain a sharper characterization of the trade-off curve \( Z(\alpha, n) \). The analysis of such cases is virtually identical to the forthcoming analysis.

![Fig. 3: The figure provides an illustration of the inherent trade-off between the price of anarchy and price of stability. First, note that the gray region is not achievable since \( 0 \leq \text{PoA}(G_f) \leq \text{PoS}(G_f) \leq 1 \) by definition. Theorem 3.1 demonstrates that the red region is also not achievable. That is, there does not exist a distribution rule achieving joint price of anarchy and price of stability guarantees in the red region. For example, if the desired price of anarchy is \( \alpha \leq 1/2 \), then a price of stability of 1 is attainable while meeting this price of anarchy demand. However, if the desired price of anarchy is \( \alpha = 1 - 1/e \), then a price of stability of 1 is no longer attainable. In fact, the best attainable price of stability is now also \( 1 - 1/e \). The distribution rule values are specified for the case when \( n = 10 \). The trade-off curve \( Z(\alpha, n) \) is illustrated for large \( n \).]

IV. USING INFORMATION TO BREACH THE ANARCHY / STABILITY FRONTIER

The previous section highlights a fundamental tension between the price of stability and price of anarchy for the given covering problem when restricted to local agent objective functions of the form (2). In this section, we challenge the role of locality in these fundamental trade-offs. That is, we show how to move beyond the price of anarchy / price of stability frontier given in Theorem 3.1 if we allow the agents to condition their choice on a higher degree of system-level information. To show this trade-off we restrict our attention to single-selection covering games.
With this goal in mind, we introduce a minimal and easily attainable piece of system-level information that can permit the realization of decision-making rules with efficiency guarantees beyond this frontier. To that end, for each allocation \( a \in A \) we define the information flow graph \((V, E)\) where each node of the graph represents an agent and we construct a directed edge \( i \rightarrow j \) if \( a_i \in A_j \) for \( i \neq j \) (no self loops). Based on this allocation-dependent graph, we define for each agent \( i \) the set \( N_i(a) \subseteq N \) consisting of all the agents that can reach \( i \) through a path in the graph \((V, E)\). Similarly, for each agent \( i \) we define

\[
Q_i(a) = \left( \bigcup_{j \in N_i(a)} A_j \right) \cup A_i,
\]

which consists of the union of \( A_i \) and all the sets of other agents that can reach \( i \) through a path in the graph. An example is shown in Figure 4. Building upon this graph we define the following quantities:

\[
V_i(a) = \max_{r \in A_i \setminus a_i} v_r, \quad (15)
\]

\[
x_i(a) = \max_{r \in Q_i(a) \setminus a_i} v_r, \quad (16)
\]

The term \( V_i(a) \) captures the highest valued resource in agent \( i \)’s choice set \( A_i \) that is not covered by any agent. If the set \( A_i \setminus a_i \) is empty, we set \( V_i(a) = 0 \). Similarly, the term \( x_i(a) \) captures the highest-valued resource in the enlarged set \( Q_i(a) \) not currently covered by any other agent. If the set \( Q_i(a) \setminus a_i \) is empty, we set \( x_i(a) = 0 \).

We are now ready to specify the information based covering game with a set of agents \( N \) and each agent has an action set \( A_i \subseteq R \). Here, we consider a state-based distribution rule that toggles between the two extreme optimal distribution rules \( f^* \) and \( f^{mc} \). More formally, the distribution rule for agent \( i \) is now of the form

\[
f_i^{sb}(a_i = r, a_{-i}) = \begin{cases} f^{mc}(|a_r|) & \text{if } V_i(a) \geq x_i(a), \\ f^*(|a_r|) & \text{otherwise,} \end{cases} \quad (17)
\]

and the corresponding utilities are given by

\[
U_i(a_i = r, a_{-i}) = v_r \cdot f_i^{sb}(a),
\]

as we allow the system-level information \( x_i(a) \) and \( V_i(a) \) to prescribe which distribution rule each agent applies. We denote with \( f_i^{sb} = \{f_i^{sb}\}_{i \in N} \) the collection of distribution rules in (17) and informally refer to it as the state-based distribution rule. Throughout, we express the distribution rule as merely \( f_i^{sb}(a) \) instead of \( f_i^{sb}(x_i(a), V_i(a)) \) for brevity.

The next theorem demonstrates how \( f_i^{sb} \) attains performance guarantees beyond the price of stability / price of anarchy frontier established in Theorem 3.1.

Theorem 4.1: Consider any single-selection maximum coverage game with a state-based distribution rule \( f_i^{sb} \) as defined above. First, an equilibrium is guaranteed to exist in any game \( G \in G_{fsb} \). Moreover, the price of anarchy and price of stability associated with the induced family of games \( G_{fsb} \) is

\[
\text{PoS}(G_{fsb}) = 1,
\]

\[
\text{PoA}(G_{fsb}) = \max_f \text{PoA}(G_f) = 1 - \frac{1}{e}.
\]

Recall from Theorem 3.1 that a consequence of attaining a price of anarchy of \( 1 - 1/e \) was a price of stability also of

\[
1 - 1/e \text{ and this was achieved by } f^* \text{ defined in (12). Using the state-based rule given in (17), a system designer can achieve the optimal price of anarchy without any consequences for the price of stability. Hence, the identified piece of system-level information was crucial for moving beyond the inherited performance limitations by adhering to our notion of local information. Whether alternative forms of system-level information could move us beyond these guarantees, or achieve these guarantees with less information, is an open research question.}

V. CONCLUSIONS

How should a system operator design a networked architecture? The answer to this question is non-trivial and involves weighing the advantages and disadvantages associated with different design choices. In this paper we highlight one novel trade-off pertaining to the worst-case and best-case performance guarantees in distributed maximum coverage problems with local information. Further, we demonstrate how a system designer can move beyond these trade-offs by equipping the agents with additional information about the system. Fully realizing the potential of multiagent systems requires the pursuit of a more formal understanding of the inherent limitations and performance trade-offs associated with networked architectures. While this paper focused purely on two performance measures, other metrics of interest include convergence rates, robustness to adversaries, fairness, among others. In each of these settings, it is imperative that a system operator fully understands the role of information within these trade-offs. Only then, will a system operator be able to effectively balance the potential performance gains with the communication costs associated with propagating additional information through the system.

VI. APPENDIX

A. Proof of Theorem 3.1

Before delving into the proof of Theorem 3.1, we present two supporting results which will be essential for the forthcoming proof. Throughout the remainder of the proof we consider games with a fixed number of agents \( n \geq 2 \).
1) A Supporting Result for the Price of Anarchy: Our first lemma demonstrates that achieving a finite price of anarchy requires that the distribution rule \( f \) satisfies \( f(1) > 0 \) and is non-negative.

**Lemma 6.1:** If \( f(1) = 0 \), then \( \text{PoA}(G^n_f) = 0 \) for any \( n \geq k \geq 2 \).

**Proof:** Suppose \( f(1) = 0 \). Consider a single-selection maximum coverage problem with three resources \( \{r_1, r_2, r_3\} \) and valuations \( v_1, v_2, \) and \( v_3 = 0 \). Furthermore, suppose \( \mathcal{A}_1 = \{(r_1), (r_2)\} \) and \( \mathcal{A}_2 = \{(r_3)\} \) for all agents \( j \in \{2, \ldots, n\} \). For any resource valuations \( v_1 > v_2 \), the action profile \((r_2, r_3, \ldots, r_3)\) is an equilibrium while the action profile \((r_1, r_3, \ldots, r_3)\) is the optimal allocation. Since \( v_1 \) can be arbitrarily large relative to \( v_2 \), this proves the lemma.

Given the above lemma, throughout the remainder of the proof we will restrict attention to the class of distribution rules

\[
f \in \mathcal{F}^n = \{ f \in \mathbb{R}_{\geq 0}^n : f(1) = 1 \}.
\]

Note that restricting \( f(1) = 1 \) is without loss of generalities. Given this restriction, we now present the results given in [33], which strengthen those given in [12], that provide a tight price of anarchy for any distribution rule \( f \in \mathcal{F}^n \) over each set of games \( G^n_f \).

**Theorem 6.1 (Theorem 2, [33]):** Consider any distribution rule \( f \in \mathcal{F}^n \). The price of anarchy associated with the induced family of games \( G^n_f \) is

\[
\text{PoA}(G^n_f) = \frac{1}{1 + \chi_f^n},
\]

where

\[
\chi_f^n = \max_{j \in [n-1]} \{ (j + 1)f(j + 1) - 1, jf(j) - f(j + 1), jf(j + 1) \}.
\]

Further, the unique distribution rule maximizing the price of anarchy over the induced games \( G^n_f \) is given in (12).

2) A Supporting Result for the Price of Stability: We now shift our attention to the other performance metric of interest, the price of stability.

**Theorem 6.2:** Consider any distribution rule \( f \in \mathcal{F}^n \). The price of stability associated with the induced family of games \( G^n_f \) satisfies

\[
\text{PoS}(G^n_f) \leq \min_{1 \leq j \leq n} \left\{ \frac{1}{1 + (j - 1)f(j)} \right\}.
\]

The optimal price of stability is \( \max_f \text{PoS}(G^n_f) = 1 \), and the unique distribution rule that achieves this price of stability over the induced games \( G^n_f \) is the marginal contribution distribution rule defined in (6). Lastly, (21) is satisfied with equality for single-selection covering games with \( n \) agents.

We will prove the above price of stability result through a series of intermediate lemmas. We begin by observing that any game \( G \) in the class \( G^n_f \) is a congestion game, and thus is a potential game as introduced in [28], with a potential function \( \phi : \mathcal{A} \to \mathbb{R} \) of the form

\[
\phi(a) = \sum_{r \in \mathcal{Q}} \sum_{j=1}^{|a_r|} v_r f(j).
\]

It is well-known that an equilibrium is guaranteed to exist in any potential game [28], and one such equilibrium is the allocation that optimizes the potential function \( \phi \), i.e., \( a^{\text{ne}} \in \arg \max_{a \in \mathcal{A}} \phi(a) \). The distribution rule maximizing the price of stability, i.e., \( f^{\text{mc}} \), and the corresponding optimal value follow immediately from the fact that \( f^{\text{mc}} \) always ensures that the potential function is precisely \( W \). This completes the first part of the proof.

To prove (21), we restrict our attention to the set of single-selection covering games, where the optimal allocation is disjoint, i.e., \( a^{\text{opt}}_i \neq a^{\text{opt}}_j \), for any \( i \neq j \). We denote such games by the set \( G^n_{\mathcal{F}^n} \). Our first lemma, stated without proof for brevity, demonstrates that restricting attention to single-selection covering games where the optimal allocation is disjoint is sufficient for characterizing the price of stability in single-selection covering games.

**Lemma 6.2:** Let \( G^n_{\mathcal{F}^n} \) denote the set of \( n \)-agent single-selection maximum coverage games with the distribution rule \( f \in \mathcal{F}^n \). The set of games \( G^n_{\mathcal{F}^n} \subset G^n_f \) where the optimal allocation is disjoint satisfies \( \text{PoS}(G^n_f) = \text{PoS}(G^n_{\mathcal{F}^n}) \).

We will now proceed with a series of claims to demonstrate that \( \text{PoS}(G^n_f) \) satisfies (21) with equality. Note this immediately implies (21) since \( \text{PoS}(G^n_{\mathcal{F}^n}) \leq \text{PoS}(G^n_f) \). The central part of the proof involves focusing on the equilibrium which maximizes the potential function in (22) in the considered class of games \( G^n_f \). From this specific equilibrium, we consider a sequence of allocations taking the form \( a^{0} = a^{\text{ne}} \) and \( a^k = (a^{\text{opt}}_{i(k)}, a^{k-1}_{i(k)}) \) for all \( k \in \{1, \ldots, m\} \) where \( i(k) \) is the deviating player in the \( k \)-th profile. The selection of the deviating players \( i = \{i(1), \ldots, i(m)\} \) is chosen according to the following rules:

(i) Let \( i(1) \in \mathcal{N} \) be any arbitrary player.

(ii) For each \( k \geq 1 \), if \( a^{\text{opt}}_{i(k)} = a^{\text{ne}}(1) \) or \( a^{\text{opt}}_{i(k)} \neq a^{\text{ne}}(1) \) then the sequence is terminated.

(iii) Otherwise, let \( i(k+1) \) be any agent in the set \( \{ j \in \mathcal{N} : a^k_j = a^{\text{opt}}_{i(k)} \} \) and repeat.

**Lemma 6.3:** Define \( Q = \cup_{i \in \mathcal{I}} a^{\text{ne}}_i \) and \( \bar{Q} = \cup_{i \in \mathcal{I}} a^{\text{opt}}_i \).

Then

\[
\sum_{i \in \mathcal{I}} U_i(a^{\text{ne}}) \geq \sum_{r \in Q \setminus Q} v_r f(|a^{\text{ne}}_r|) + \sum_{r \in \bar{Q} \setminus Q} v_r.
\]

**Proof:** We begin with two observations on the above sequence of allocations: (a) the sequence of allocations can continue at most \( n \) steps due to the disjointness of \( a^{\text{opt}} \) and (b) if the sequence continues, it must be that for player \( i(k+1) = a^{\text{opt}}_{i(k)} \). Observation (b) ensures us that

\[
\psi = \sum_{k=1}^{m-1} U_i(k+1)(a^k) - U_i(k)(a^k) = 0.
\]
Accordingly, we have that
\[
\phi(a^0) - \phi(a^m) = \sum_{k=0}^{m-1} \phi(a^k) - \phi(a^{k+1}) = \sum_{k=0}^{m-1} U_i(k+1)(a^k) - U_i(k)(a^{k+1}) = U_i(1)(a^0) - U_i(m)(a^m) + \psi = U_i(1)(a^0) - U_i(m)(a^m) \geq 0.
\]

The first and third equalities follow by rearranging the terms. The second equality can be shown using the definition of $\phi$ as in (22); the last equality follows by (24). The inequality derives from the fact that $a^0 = a^{ne}$ optimizes the potential function. Thanks to observation (b), one can show that
\[
Q \setminus \bar{Q} = a_{\text{ne}}(1) \setminus a_{\text{opt}}(m) \quad \text{and} \quad \bar{Q} \setminus Q = a_{\text{opt}}(1) \setminus a_{\text{ne}}(m).
\]

If $Q \setminus \bar{Q} \neq \emptyset$, it must be that $a_{\text{ne}}(1) \neq a_{\text{opt}}(m)$ so that $Q \setminus \bar{Q} = a_{\text{ne}}(1)$ and $\bar{Q} \setminus Q = a_{\text{opt}}(m)$. Using $a_{\text{ne}}(1) \neq a_{\text{opt}}(m)$ in condition (ii), tells us that $a_{\text{opt}}(m) \notin a^{ne}$ and thus the resource $a_{\text{opt}}(m)$ is not chosen by anyone else in the allocation $a^m$. Thus, when $Q \setminus \bar{Q} \neq \emptyset$, $U_i(1)(a^0) - U_i(m)(a^m) = \sum_{r \notin Q \cap \bar{Q}} v_r f(|a^{ne}|_r) - \sum_{r \notin \bar{Q} \cap Q} v_r \geq 0.
\]

When $Q \setminus \bar{Q} = \emptyset$, also $Q \setminus Q = \emptyset$ and thus the previous inequality still holds. Rearranging the terms and adding $\sum_{r \in Q \cap \bar{Q}} v_r f(|a^{ne}|_r)$ to each side gives us
\[
\sum_{r \notin Q} v_r f(|a^{ne}|_r) \geq \sum_{r \notin Q \setminus \bar{Q}} v_r f(|a^{ne}|_r) + \sum_{r \in \bar{Q} \setminus Q} v_r.
\]

Finally note that
\[
\sum_{i \in \mathcal{I}} U_i(a^{ne}) \geq \sum_{r \notin Q} v_r f(|a^{ne}|_r)
\]

which together with (26) completes the proof.

Our third lemma shows that there exist a collection of disjoint sequences that covers all players in $N$. We will express a sequence merely by the deviating player set $\mathcal{I}$ with the understanding that this set uniquely determines the sequence of allocations.

Lemma 6.4: There exists a collection of deviating players $\mathcal{I}^1, \ldots, \mathcal{I}^p$ chosen according to the process described above such that $\cup_{i \in \mathcal{I}^i} N = N$ and $\mathcal{I}^i \cap \mathcal{I}^k = \emptyset$ for any $j \neq k$.

Proof: Suppose $\mathcal{I}^1, \mathcal{I}^2, \ldots, \mathcal{I}^k$ represent the first $k$ sequences of deviating players. Further assume that they are all disjoint. Choose some player $i \in N \setminus \cup_{k} \mathcal{I}^k$ to start the $(k + 1)$-th sequence. If no such player exists, we are done. Otherwise, construct the sequence according to the process depicted above. If the sequence terminates without selecting a player in $\cup_k \mathcal{I}^k$, then repeat this process to generate the $(k + 2)$-th sequence. Otherwise, let $i^{k+1}(j), j \geq 2$, denote the first player in the $(k + 1)$-th sequence contained in the set $\cup_k \mathcal{I}^k$. Since $a_{\text{opt}}^i \neq a_{\text{opt}}^j$ (for $i \neq j$), this player must be contained in the set $\cup_k \mathcal{I}^k(1)$, i.e., the first player in a previous sequence. Suppose this player is $i^l(1)$, where $l \in \{1, \ldots, k\}$. If this is the case, replace the $l$-th sequence with $k^{l+1}(1), i^{k+1}(j - 1), \mathcal{I}^j$ which is a valid sequence and disjoint from the others. Then repeat the process above to choose the $(k + 1)$-th sequence. Note that this process can continue at most $n$-steps and will always result in a collection of disjoint sequences that cover all players in $N$. This completes the proof.

In the following we complete the proof of Theorem 6.2, by means of Lemmas 6.3 and 6.4.

Proof: We are showing a lower bound on the price of stability. Let $\mathcal{I}_1, \ldots, \mathcal{I}_p$ denote a collection of deviating players that satisfies Lemma 6.4. Further, let $Q^k$ and $\bar{Q}^k$ be defined as above for each sequence $k = 1, \ldots, p$. Using the result (23) from Lemma 6.3, we have
\[
\sum_{i \in N} U_i(a^{ne}) = \sum_{k=1}^{p} \sum_{i \in \mathcal{I}^k} U_i(a^{ne}) \geq \sum_{r \in a^{ne} \setminus a} v_r f(|a^{ne}|_r) + \sum_{r \in a^{ne} \setminus a} v_r,
\]

where the above equality follows from the fact that $\bar{Q}^k \cap \bar{Q}^j = \emptyset$ for any $i \neq j$ which is due to the disjointness of $a^{ne}$. Using the definition of $U_i(a^{ne})$, we have
\[
\sum_{r \in a^{ne} \setminus a} v_r f(|a^{ne}|_r) + \sum_{r \in a^{ne} \setminus a} v_r \geq \sum_{r \in a^{ne} \setminus a} v_r.
\]

Define $\gamma = \max_{j \leq n} (j - 1) f(j)$. Working with the above expression we have
\[
\sum_{r \in a^{ne} \setminus a} v_r (\gamma + 1) + \sum_{r \in a^{ne} \setminus a} v_r \gamma \geq \sum_{r \in a^{ne} \setminus a} v_r,
\]

which gives us that
\[
(\gamma + 1) W(a^{ne}) \geq W(a^{opt})
\]

which completes the lower bound.

We will now provide an accompanying upper bound on the price of stability. To that end, consider a family of examples parameterized by a coefficient $j \in \{1, \ldots, n\}$. For each $j$, the game consists of $j$ agents and $(j + 1)$-resources $R = \{r_0, r_1, \ldots, r_j\}$ where the values of the resources are $v_{r_0} = 1$ and $v_{r_1} = \cdots = v_{r_j} = f(j) - \epsilon$ where $\epsilon > 0$ is an arbitrarily small constant, and the action set of each agent $i \in \{1, \ldots, j\}$ is $A_i = \{r_0, r_1\}$. The unique equilibrium is of the form $a^{ne} = (r_0, \ldots, r_0)$ as every agent selects resource $r_0$ and the total welfare is $W(a^{ne}) = 1$. The optimal allocation is of the form $a^{opt} = (r_0, r_2, \ldots, r_j)$ which generates a total welfare of $W(a^{opt}) = 1 + (j - 1)(f(j) - \epsilon)$. Performing a worst case analysis over $\epsilon$ and $j$ gives (21), which completes the proof.

3) Proof of Theorem 3.1: Part (i) of Theorem 3.1 follows immediately from Theorem 6.1. The remaining parts follow from the following enriched version of Theorem 3.1. Before stating this enriched version, we denote with $\mathcal{I}^\alpha$, $\alpha \in (0, 1]$, the family of distribution rules that guarantee a price of
anarchy of at least \( \alpha \), i.e.,
\[
\mathcal{F}_n^\alpha = \{ f \in \mathcal{F}_n^\alpha : \text{PoA}(G_f^j) \geq \alpha \}.
\]

Additionally, we let
\[
\text{PoS}(G^\alpha_n; \alpha) = \max_{f \in \mathcal{F}_n^\alpha} \text{PoS}(G_f^j)
\]
be the best achievable price of stability given that the price of anarchy is guaranteed to exceed \( \alpha \), where \( \alpha \in (0, 1) \).

**Theorem 6.3:** The function \( \text{PoS}(G^\alpha_n; \alpha) \) satisfies

i) For any \( \alpha \leq 1/2 \), \( \text{PoS}(G^\alpha_n; \alpha) = 1 \).

ii) For any \( \frac{1}{2} < \alpha \leq \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \)
\[
\text{PoS}(G^\alpha_n; \alpha) = \frac{1}{1 + \xi_n^\alpha} \sum_{j=1}^{n-1} j! \left( 1 - \left( \frac{1}{\alpha} - 1 \right) \sum_{i=1}^{j} \frac{1}{i!} \right)
\]
for single-selection covering games.

iii) Lastly, if \( \alpha = \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \) the
\[
\text{PoS}(G^\alpha_n; \alpha) = \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j)
\]
which converges to \( 1 - 1/e \) as \( n \to \infty \).

**Proof:** i) Consider the marginal contribution distribution rule \( f_{mc} \), defined in (6). It is straightforward to verify that \( \chi_{f_{mc}}^n = 1 \) and hence \( f_{mc} \in \mathcal{F}_n^\alpha \) for any \( \alpha \leq 1/2 \). Further, it is well-known that \( \text{PoS}(G^f_{mc}) = 1 \), see [27]. This completes the first part of the proof.

ii) Restricting our attention to single-selection covering games, from Theorem 6.2 and (27) we have
\[
\text{PoS}(G^\alpha_n; \alpha) = \max_{f \in \mathcal{F}_n^\alpha} \min_{1 \leq j \leq n} \left\{ \frac{1}{1 + (j - 1)f(j)} \right\}
\]
or equivalently
\[
\text{PoS}(G^\alpha_n; \alpha) = \max_{f \in \mathcal{F}_n^\alpha} \min_{1 \leq j \leq n} \left( \frac{1}{1 + jf(j)} \right),
\]
for all other \( 2 \leq j \leq n \). Thanks to Theorem 6.1, the constraint \( \text{PoA}(G_f^j) \geq 1 \) can be written as
\[
\begin{align*}
    jf(j) - \left( \frac{1}{\alpha} - 1 \right) & \leq f(j + 1) \quad 1 \leq j \leq n - 1, \\
    jf(j + 1) & \leq \left( \frac{1}{\alpha} - 1 \right) \quad 1 \leq j \leq n - 1, \\
    (j + 1)f(j + 1) - 1 & \leq \left( \frac{1}{\alpha} - 1 \right) \quad 1 \leq j \leq n - 1.
\end{align*}
\]
Thus, the optimization problem in (30) is equivalent to
\[
\begin{align*}
    \min_{f \in \mathcal{F}_n^\alpha} \max_{1 \leq j \leq n - 1} jf(j + 1) \\
    \text{s.t.} \quad jf(j) - \left( \frac{1}{\alpha} - 1 \right) & \leq f(j + 1) \quad 1 \leq j \leq n - 1, \\
    jf(j + 1) & \leq \left( \frac{1}{\alpha} - 1 \right) \quad 1 \leq j \leq n - 1, \\
    (j + 1)f(j + 1) & \leq \left( \frac{1}{\alpha} - 1 \right) \quad 1 \leq j \leq n - 1, \\
    f(j) & \geq 0 \quad 1 \leq j \leq n, \\
    f(1) & = 1.
\end{align*}
\]
Recall that \( f(1) = 1 \). Recursively applying the first set of inequalities, it immediately follows that for every \( j \) with \( 1 \leq j \leq n - 1 \)
\[
\begin{align*}
    jf(j + 1) & \geq jf(j)(1 - \left( \frac{1}{\alpha} - 1 \right)\sum_{i=1}^{j} \frac{1}{i!}.
\end{align*}
\]
Since our objective is precisely to minimize the quantity \( \max_{1 \leq j \leq n - 1} jf(j + 1) \), we find a candidate solution by first solving for the distribution rule that satisfies the \( n - 1 \) linear inequalities with equality, or set it to zero if the term \( jf(j)(1 - 1/\alpha - 1) \) is negative. Such a distribution rule can be computed recursively, and is of the form
\[
\begin{align*}
    \hat{f}(j) = \max_{(j-1)!} \left( 1 - \left( \frac{1}{\alpha} - 1 \right) \sum_{i=1}^{j} \frac{1}{i!} \right), 0.
\end{align*}
\]
By construction, the above distribution satisfies \( \hat{f}(1) = 1 \) as well as the first and fourth set of constraints in the optimization problem above. In the following we verify that \( \hat{f} \) also satisfies the remaining set of constraints, and thus is optimal.

We begin with the second set of constraints, i.e., we wish to show \( jf(j + 1) \leq (1/\alpha - 1) \) for all \( \alpha \) with \( \frac{1}{2} < \alpha \leq \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \) and for all \( j \) with \( 1 \leq j \leq n - 1 \). This is equivalent to
\[
\begin{align*}
    jf(j)(1 - \left( \frac{1}{\alpha} - 1 \right)\sum_{i=1}^{j} \frac{1}{i!} \) \leq \frac{1}{\alpha} - 1, \\
    \text{since the result follows immediately when } \hat{f} \text{ takes the value } \hat{f}(j) = 0. \text{ After some manipulation this is equivalent to showing}
\end{align*}
\]
\[
\begin{align*}
    \frac{1}{j!} + \sum_{i=1}^{j} \frac{1}{i!} \geq \left( \frac{1}{\alpha} - 1 \right)^{-1}
\end{align*}
\]
for all \( \alpha \) with \( \frac{1}{2} < \alpha \leq \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \) and for all \( j \) with \( 1 \leq j \leq n - 1 \). Since \( \alpha \leq \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \), the result follows if we are able to prove the above inequality for \( \alpha = \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) \), i.e.,
\[
\begin{align*}
    \frac{1}{j!} + \sum_{i=1}^{j} \frac{1}{i!} \geq \left( \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) - 1 \right)^{-1}.
\end{align*}
\]
The expression for the optimal price of anarchy was found [12, Corollary 1], and amounts to
\[
\begin{align*}
    \max_{f \in \mathcal{F}_n^\alpha} \text{PoA}(G_f^j) = 1 - \frac{1}{(n-1)(n-1)} + \sum_{i=0}^{n-1} \frac{1}{i!}.
\end{align*}
\]
Repeating this in (31), we are left to prove that for all \(1 \leq j \leq n-1\)
\[
\frac{1}{j!} + \sum_{i=0}^{j} \frac{1}{i!} \geq \frac{1}{(n-1)(n-1)!} + \sum_{i=0}^{n-1} \frac{1}{i!}.
\]
(32)
To do so, let us define the function \(g : \mathbb{N} \to \mathbb{R}\) as
\[
g(j) = \frac{1}{j!} + \sum_{i=0}^{j} \frac{1}{i!},
\]
and observe that \(g(j)\) is non-increasing in \(j\). To see this note that
\[
g(j) \geq g(j+1) \iff \frac{1}{j!} + \sum_{i=0}^{j} \frac{1}{i!} \geq \frac{1}{(j+1)(j+1)!} + \sum_{i=0}^{j+1} \frac{1}{i!} \iff \frac{1}{j+1}(j+1)! \geq \frac{1}{j+1},
\]
which holds for any \(j \in \mathbb{N}\). Thus, we conclude that for \(j\) with \(1 \leq j \leq n-1\) it holds
\[
g(j) \geq g(n-1),
\]
which is precisely the desired result in (32), thus proving that the third set of inequalities is satisfied.

We now turn our attention to the third set of constraints, and wish to prove that \((j+1)f(j+1) \leq 1/\alpha\) for all \(\alpha\) with \(1/2 < \alpha \leq \max_{f \in \mathcal{F}} \text{PoA}(G^n_f)\) and for all \(j\) with \(1 \leq j \leq n-1\). This follows directly as a consequence of \(j f(j+1) \leq (1/\alpha - 1)\) previously shown. Indeed
\[
j f(j+1) \leq \frac{1}{\alpha - 1} \iff f(j+1) \leq \frac{1}{j} \left( \frac{1}{\alpha - 1} \right),
\]
which implies the desired result
\[
(j+1)f(j+1) \leq \frac{1}{\alpha - 1} \left( \frac{1}{\alpha - 1} \right) = \left( \frac{1}{\alpha} - 1 \right) + \frac{1}{j} \left( \frac{1}{\alpha - 1} \right) \leq \frac{1}{\alpha - 1} + 1 = \frac{1}{\alpha},
\]
where the last inequality follows from \(j \geq 1\) and \(1/\alpha - 1 \leq 1\) (due to \(\alpha > 1/2\)). This concludes the above reasoning, and shows that \(f\) is a solution to (30). Using Theorem 6.2 on such distribution rules gives the desired result in (28).

iii) The final result can be proven as follows: replace \(\alpha\) in (28) with the optimal price of anarchy from [12, 33], i.e., set
\[
\alpha = 1 - \frac{1}{(n-1)(n-1)!} + \sum_{i=0}^{n-1} \frac{1}{i!}.
\]
After some manipulation,
\[
\text{PoS}(G^n; \alpha) = \frac{1}{(n-1)(n-1)!} + \sum_{i=0}^{n-1} \frac{1}{i!} = \max_{f \in \mathcal{F}^n} \text{PoA}(G^n_f),
\]
proving the claim in (29). Taking the limit \(n \to \infty\) in the previous expression, gives the final result.

B. Proof of Theorem 4.1

We will now present the proof of Theorem 4.1. For readability, we will split the proof into the following two subsections focusing on the price of anarchy and price of stability respectively.

1) Proof of PoS Result: We begin our proof with a lemma that identifies an important structure regarding the state based distribution rule \(\{f_i\}\).

Lemma 6.5: Let \(a\) be any allocation. Then for each agent \(i \in N\), one of the following two statements is true.
- \(U_i(a'_i = r, a_{-i}) = v_r \cdot f_nmc(|a_{-i}| + 1),\forall a'_i \in \mathcal{A}_i\); or
- \(U_i(a'_i = r, a_{-i}) = v_r \cdot f^*(|a_{-i}| + 1),\forall a'_i \in \mathcal{A}_i\).

Informally, this lemma states that for a given allocation \(a\), the state based distribution rule will either evaluate every resource at \(f^*\) or \(f^{mc}\) for a given agent \(i\).

Proof: Let \(a\) be any allocation and \(i\) be any agent. Extend the definition of \(x_i(a)\) in (16) as
\[
y_i(a) = \max_{r \in (Q_i(a),a) \cap \mathcal{A}_i} v_r,
\]
\[
z_i(a) = \max_{r \in (Q_i(a),a) \cap \mathcal{A}_i} v_r,
\]
and note that \(x_i(a) = \max\{y_i(a), z_i(a)\}\). First observe that for any \(a'_i \in \mathcal{A}_i\): (i) \(y_i(a) \leq V_i(a)\); (ii) \(z_i(a) = z_i(a'_i, a_{-i})\); and (iii) \(V_i(a) = V_i(a'_i, a_{-i})\). Accordingly, \(x_i(a) > V_i(a)\) if and only if \(x_i(a) = z_i(a)\). Consequently, \(x_i(a) > V_i(a)\) if and only if \(x_i(a'_i, a_{-i}) > V_i(a'_i, a_{-i})\) which completes the proof.

We will now prove that an equilibrium exists and the price of stability is 1. In particular, we will show that the optimal allocation \(a^{opt}\) is in fact an equilibrium.

Proof: Let \(a^{opt}\) be an optimal allocation. We begin by showing that \(V_i(a^{opt}) \geq x_i(a^{opt})\) for all \(i \in N\). Suppose this was not the case, and there exists an agent \(i\) such that \(V_i(a^{opt}) < x_i(a^{opt})\). This implies that there exists a resource \(r \in Q_i(a^{opt})\) such that \(v_r > v_r\). By definition of \(Q_i(a)\) and \(N_i(a)\) there exists a sequence of players \(\{i_0, i_1, \ldots, i_m\}\) such that \(a^{opt}_{i} \in A_{i_{k+1}}\) for all \(k \geq 1\), and \(r \in A_{i_m}\). Hence, consider a new allocation where \(a^{opt}_{i} = a^{opt}_{i_{k+1}}\) for all \(k \in \{0, \ldots, m-1\}\) and \(a^{opt}_{i_m} = r\), where \(a^{opt}_{j} = a^{opt}_{j}\) for all other agents \(j \notin \{i_0, i_1, \ldots, i_m\}\). The welfare of this allocation is \(W(a) \geq W(a^{opt}) + v_r - v_r > W(a^{opt})\), which contradicts the optimality of \(a^{opt}\). This means that every agent will be using the marginal cost distribution rule to evaluate its utility in the allocation \(a^{opt}\).

Now, suppose \(a^{opt}\) is not an equilibrium for sake of contradiction. This means, that there exists an agent \(i\) with an action \(a_{i} \in \mathcal{A}\) such that \(U_i(a_{i}, a^{opt}_{-i}) > U_i(a^{opt}_{i}, a^{opt}_{-i})\). Since agents are using the marginal contribution distribution rule, which follows from \(V_i(a^{opt}) \geq x_i(a^{opt})\), we have
\[
W(a_{i}, a^{opt}_{-i}) = W(a^{opt}) - U_i(a^{opt}_{i}, a^{opt}_{-i}) + U_i(a_{i}, a^{opt}_{-i}) > W(a^{opt}),
\]
which contradicts the optimality of \(a^{opt}\). This completes the proof.

2) Informal Discussion of PoA Result: In the following we give an informal discussion for the price of anarchy result. Consider a game \(G = (N, \mathcal{R}, \{\mathcal{A}_i\}, \{f_i\}, \{v_r\})\) with
Cardinality bounded by \(k\). Let \(a^{ne}\) be any equilibrium of the game \(G\). A crucial part of the forthcoming analysis will center on a new game \(G'\) derived from the original game \(G\) and the equilibrium \(a^{ne}\), i.e.,

\[(G, a^{ne}) \rightarrow G'.\]

This new game \(G'\) possesses the identical player set, resource set, and valuations of the resources as the game \(G\). The difference between the games are (i) the action sets and (ii) the new game \(G'\) employs the Gairing distribution rule, \(f^*\), as opposed to the state-based distribution rule \(f^{sb}\). Informally, the proof proceeds in the following two steps:

- **Step 1**: We prove that the equilibrium \(a^{ne}\) of \(G\) is also an equilibrium of \(G'\). Since the player set, resource set, and valuations of the resources are unchanged we have that

\[W(a^{ne}; G') = W(a^{ne}; G),\]

where we write the notation \(W(a^{ne}; G')\) to mean the welfare accrued at the allocation \(a^{ne}\) in the game \(G'\).

- **Step 2**: We show that the optimal allocation \(a^{opt}\) in the new game \(G'\) is at least as good as the optimal allocation in the original game \(G'\), i.e.,

\[W(a^{opt}; G') \geq W(a^{opt}; G').\]

Combining the results from Step 1 and Step 2 give us

\[\frac{W(a^{ne}; G)}{W(a^{opt}; G')} \geq \frac{W(a^{ne}; G')}{W(a^{opt}; G')} \geq \text{PoA}(G_f),\]

where the last inequality follows from Theorem 6.1 since \(G'\) employs \(f^*\).

**3) Proof of PoA Result**: In this section we present the formal proof for the price of anarchy result pertaining to the state based design. We begin by providing the formal details on the game construction highlighted above.

**Construction of game \(G'\)**: We will now provide the construction of the new game \(G'\) from the game \(G\) and the equilibrium \(a^{ne}\). We begin with some notation that we will use to construct the new agents’ action sets in the game \(G'\).

For each \(i \in N\), let \(r_i = a^{ne}_i\) and define

\[H_i = \{r \in A_i : v_r f^* (|a^{ne}_r|) < v_r f^* (1 + |a^{ne}_r|)\},\]

(35)

to be the set of resources that will give a strictly better payoff to agent \(i\) if the agent used the marginal contribution distribution rule everywhere. Accordingly, we have

\[U_i(a^{ne}_i; G') = \max_{r \in A_i \setminus a^{ne}_i} v_r = V_i(a^{ne}_i) \geq x_i(a^{ne}_i),\]

(39)

where the first equality follows from the equilibrium conditions coupled with the use of the marginal contribution distribution rule and the inequality follows from the use of the marginal contribution distribution rule and (17). We will conclude the proof by a case study on the potential values of \(U_i(a^{ne}_i, a^{ne}_j; G)\). For brevity in the forthcoming arguments we let \(r = a^{ne}_i\).

- Case (i): Suppose \(U_i(a^{ne}_i, a^{ne}_j; G) = 0\). In this case, we have that \(x_i(a^{ne}_i) = 0\) from (39) which implies that any resource \(r' \in B_i\) has value \(v_{r'} = 0\). Hence, \(U_i(a^{ne}_i, a^{ne}_j; G') = U_i(r', a^{ne}_j; G') = 0\) and we are done.

- Case (ii): Suppose \(U_i(a^{ne}_i, a^{ne}_j; G) > 0\). Based on the definition of the marginal cost distribution rule, this implies that \(|a^{ne}_r| = 1\), and hence

\[U_i(a^{ne}_i; G) = v_r f^{mc}(|a^{ne}_r|) = v_r f^*(|a^{ne}_r|) = U_i(a^{ne}_i; G'),\]

which follows from the fact that \(f^{mc}(1) = f^*(1) = 1\). Hence, \(U_i(a^{ne}_i; G') = v_{r'}\). For any \(r' \in B_i\), we have

\[U_i(r', a^{ne}_j; G') = v_{r'} f^*(1 + |a^{ne}_r'|) = v_{r'} \leq x_i(a^{ne}_i),\]

where the last inequality follows for the definition of \(x_i(a^{ne}_i)\). Combining with (39) gives us

\[U_i(a^{ne}_i, a^{ne}_j; G') \geq U_i(r', a^{ne}_j; G'),\]

which completes the proof.

**Formal Proof of Step 2**: We begin with a lemma that highlights a structure associated with the action sets \(\{A'_i\}\) in the new constructed game \(G'\).

**Lemma 6.6**: If \(r \in A'_i \setminus A_i\) for some agent \(i \in N\) and resource \(r \in R\), then there exists an agent \(j \neq i\) such that \(a^{ne}_j = r\) and consequently \(r \in A'_j\).
Proof: Suppose \( r \in A'_i \setminus A_i \) for some agent \( i \in N \) and resource \( r \in R \). Then, \( r \in H_i \) by definition of the set \( A'_i \). By Lemma 6.5, each agent must either be a marginal contribution agent, i.e., uses \( f^n \) at all resources, or a Gairing agent, i.e., uses \( f^* \) at all resources. Since, \( i \in I \) and \( a^m \) is an equilibrium, agent \( i \) must be a marginal contribution agent, i.e.,
\[
U_r(a^n) = v_r \cdot f^n(\|a^n\|_r) = v_r \cdot f^n(\|a^m\|_r + 1),
\]
where the inequality follows from the equilibrium conditions. Suppose by contradiction that \( \|a^n\|_r = 0 \). In this case we have
\[
v_r \cdot f^n(\|a^n\|_r + 1) = v_r \cdot f^*(\|a^n\|_r + 1).
\]
which follows from the fact that \( f^n(1) = f^*(1) = 1 \). Since \( f^*(k) \geq f^n(k) \) for all \( k \geq 1 \), this implies
\[
v_r \cdot f^*(\|a^n\|_r) \geq v_r \cdot f^n(\|a^n\|_r) \geq v_r \cdot f^*(\|a^n\|_r + 1).
\]
Hence, \( r \notin H_i \) leading to the contradiction. This completes the proof.

We exploit the result of Lemma 6.6 to prove Step 2.

Proof: We conclude the proof by constructing an allocation \( a \in A' \) that satisfies \( W(a; G') = W(a^{opt}; G) \) where \( a^{opt} \in \text{arg max}_{a \in A} W(a; G) \). We begin with an initial allocation \( a \) where for each agent \( i \in N \)
\[
a = \begin{cases} 
  a_i^{opt} & \text{if } a_i^{opt} \in A'_i, \\
  \emptyset & \text{else,}
\end{cases}
\]
That is, we assign each agent the agent’s optimal allocation choice if it is available to them in the new action set \( A'_i \). If all agents received their optimal choice, then the proof is complete.

If this is not the case, then there will be a set of uncovered resources \( U(a) = \{ r \in a^{opt} : |a_r| = 0 \} \) which we denote by \( U(a) = \{ r_1, \ldots, r_m \} \). We will now argue that we can construct a new allocation \( a' \) that covers one additional resource from the set \( U(a) \), i.e., \( |U(a')| = |U(a)| + 1 \) and \( a \subseteq a' \), where we denote with \( |U(a)| \) the cardinality of \( U(a) \).

To that end, consider any uncovered resource \( r_0 \in U(a) \). By definition, there exists an agent \( i_0 \in N \) such that \( a_{i_0} = 0 \) but \( r_0 \notin A_{i_0}' \). Consequently, we have that \( r_0 \in H_{i_0} \) and by Lemma 6.6 we know that there exists an agent \( i_1 \neq i_0 \) such that \( a_{i_1} = 0 \). Since \( a_{i_1}^{opt} = r_0 \) we also have that \( r_0 \in A_{i_1}' \) by definition. We now analyze the following three cases:

- Case 1: Suppose \( a_{i_1} = \emptyset \). Then define a new allocation \( a'_i = [r_0, a'_j] \) for all \( j \neq i_1 \), and we are done.

- Case 2: Suppose \( a_{i_1} = r_1 \) and \( a_{i_1}^{opt} = 0 \), meaning that there are no agents at the resource \( r_1 \) in the equilibrium allocation. Then by definition \( r_1 \in B_{i_0} \) and \( r_1 \in A_{i_0}' \). Define the allocation \( a'_i = [r_1, a_{i_1}'] = r_0, a'_j = a_j \) for all \( j \neq i_1, i_0 \) and we are done.

- Case 3: Suppose \( a_{i_1} = r_1 \) and \( a_{i_1}^{opt} = 0 > 0 \), meaning that there are agents at the resource \( r_1 \) in the equilibrium allocation. Select any agent \( i_2 \) such that \( a_{i_2} = r_1 \).

1. If \( a_{i_2} = \emptyset \), then consider the allocation \( a'_i = [r_0, a'_{i_2} = r_1] \) and \( a'_j = a_j \) for all \( j \neq i_1, i_2 \) and we are done.

2. Otherwise, if \( a_{i_2} = r_2 \), then let \( a_{i_1} = r_0, a'_{i_2} = r_{1}, \) and repeat Case 2 or Case 3 depending on whether \( |a_{i_1}^{opt}|_{r_2} > 0 \). Note that Case 3-(ii) can be repeated at most \( n \) iterations until an alternative case that terminates is reached. To see this, note that each time an agent is given a new choice in this process, i.e., \( a_i = a_i' \neq a_i \), the agent’s new choice is the agent’s equilibrium choice, i.e., \( a_i' = a_{i}^{opt} \). Therefore, once an agent is assigned a new choice, the agent will never be reassigned in this process.

Starting from \( a \) as defined above, the above process results in a new allocation \( a' \) that satisfies \( |U(a')| = |U(a)| + 1 \) and \( a \subseteq a' \). As with the allocation \( a \), the allocation \( a' \) satisfies \( a_{i}^{opt} = 1 \) and \( a' \subseteq a^{opt} \). If \( a' = a^{opt} \), we are done. Otherwise, we can repeat the process depicted above to generate a new allocation \( a'' \) such that \( |U(a'')| = |U(a')| - 1 \) as nowhere in the process did we rely on the fact that \( a_i = a_i^{opt} \). Repeating these arguments recursively provides the result.

\[\square\]

References


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