Tight Bounds on the Ratio of Network Diameter to Average Internode Distance

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About This Presentation

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<td>First</td>
<td>Fall 2018</td>
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Network Attributes

- Diameter: \( D \)
- Average distance: \( D \)
- Node degree: \( d \) (max, min)
- Bisection bandwidth: \( B \)
- Longest wire
- Heterogeneous or homogeneous nodes

Other attributes:
- Regularity
- Scalability
- Packageability
- Robustness

Number of nodes: \( p \)
Distances in Path and Mesh Networks

$D_{p\text{-path}} = p - 1$

$\Delta_{p\text{-path}} = (1/p^2)\sum_{0 \leq j \leq p-1} [\sum_{0 \leq i \leq j} (j - i) + \sum_{j \leq i \leq p-1} (i - j)]$

$\Delta_{p\text{-path}} = (1/p^2)\sum_{0 \leq j \leq p-1} [j(j + 1) - j(j + 1)/2$

$+ (p - j)(p - 1 + j)/2 - j(p - j)] = (1/3)(p - 1/p)$

$D_{q\text{D-mesh}} = \sum_{1 \leq i \leq q} n_i - q$

$\Delta_{q\text{D-mesh}} = (1/3)[\sum_{1 \leq i \leq q} (n_i - 1/n_i)]$

$D_{p\text{-path}} / \Delta_{p\text{-path}} \approx 3$

$D_{q\text{D-mesh}} / \Delta_{q\text{D-mesh}} \approx 3$
Distances in Ring and Torus Networks

\[
D_{p\text{-ring}} = \left(\frac{1}{2}\right)\left[p - \left(p \mod 2\right)/p\right]
\]
\[
\Delta_{p\text{-ring}} = \left(\frac{1}{4}\right)\left[p - \left(p \mod 2\right)/p\right]
\]
\[
D_{q\text{D-torus}} = \left(\frac{1}{2}\right)\sum_{1 \leq i \leq q} \left[n_i - \left(n_i \mod 2\right)/n_i\right]
\]
\[
\Delta_{q\text{D-torus}} = \left(\frac{1}{4}\right)\sum_{1 \leq i \leq q} \left[n_i - \left(n_i \mod 2\right)/n_i\right]
\]

Alternative formula: \( D_{p\text{-ring}} = \left\lceil \left(\frac{p - 1}{2}\right) \right\rceil \)

\[
\frac{D_{p\text{-ring}}}{\Delta_{p\text{-ring}}} = 2
\]
\[
\frac{D_{q\text{D-torus}}}{\Delta_{q\text{D-torus}}} = 2
\]
Distances in Complete Binary Trees (1)

\[ D_{\text{binary-tree}} = 2l - 2 = 2 \log_2 m - 2 \]

[Let \( m = 2^l \); \( T_m \) has \( 2^l - 1 \) nodes]

\[ \sigma(T_m) = 1 \times 2^1 + 2 \times 2^2 + \ldots + (l - 1) \times 2^{l-1} = (l - 2)2^l + 2 \]

\[ = m \log_2 m - 2m + 2 \]

\( S(L, L) = S(R, R) = S(T_{m/2}) \)

\( S(r, L) = S(r, R) = S(L, r) = S(R, r) \)

\[ = m/2 - 1 + \sigma(m/2) \]

\( S(L, R) = S(R, L) \)

\[ = (m/2 - 1)^2 \left[ 2 + 2\sigma(m/2)/(m/2 - 1) \right] \]

\[ = (m - 2)\sigma(m/2) + (m - 2)^2/2 \]
Distances in Complete Binary Trees (2)

\[ S(T_m) = 2S(L, L) + 4S(r, L) + 2S(L, R) \]
\[ = 2S(T_{m/2}) + m^2 \log_2 m - 2m^2 + 2m \]
\[ = 2m^2 \log_2 m - 6m^2 + 2m \log_2 m + 6m \]

\[ \Delta(T_m) = \frac{(2m^2 \log_2 m - 6m^2 + 2m \log_2 m + 6m)/(m-1)^2}{\text{Asymptotic value}} \]

Recall \[ D(T_m) = 2l - 2 = 2 \log_2 m - 2 \]

\[ \lim_{m \to \infty} \Delta(T_m) = D(T_m) - 4 \]
\[ \lim_{m \to \infty} \frac{D(T_m)}{\Delta(T_m)} = 1 \]
Incomplete and Balanced Binary Trees

Complete binary tree: All \(2^{l-1} = (p + 1)/2\) leaves are at level \(l\)

Incomplete binary tree: There are leaves in 2 or more levels

Balanced binary tree: Leaves are at levels \(l\) and \(l-1\)

Complete binary tree: All leaves are at level \(l\)

\[
p = 2^l - 1 \quad p < 2^l - 1 \quad 2^{l-1} - 1 < p < 2^l - 1
\]
Distances in Balanced Binary Trees

Theorem 1: In an incomplete binary tree with more than one incomplete level, removing a node from an incomplete level $k$ and adding a node to an incomplete level $k - j$ ($j > 0$) does not increase the diameter and always reduces the average internode distance. ■

Theorem 2: In a balanced binary tree, with the final level $l$ containing missing nodes in both subtrees, removing a node from a side with equal or fewer nodes and adding a node to the other side decreases the average internode distance, with no increase in diameter. ■
Extremes in Distance Ratio Bounds

\[ D(G) = m \]
\[ \Delta(G) = \left[ n^2 + m(m^2-1)/3 + 2(n-1)(2+3+\ldots+m) \right]/(n+m-1)^2 \]

\[ \lim_{n \to \infty} \Delta(G) = 1 \]

\[ \lim_{n \to \infty} D(T_m) / \Delta(T_m) = m \]

So, we can make the \( D/\Delta \) ratio as close as we want to the arbitrary value \( m \)

The extreme graph \( G \)
Ratio Bounds in Symmetric Networks

Theorem 3: Given a node-symmetric network with node degree $d$, diameter $D$, and average internode distance $\Delta$, we have $D/2 \leq \Delta \leq D$. ■

Proof outline: Consider a node $X$ and a diametrically opposite node to it, $Y$. Let there be $d$ nodes that are distance-1 to $X$ (its immediate neighbors). By node-symmetry, $Y$ also has $d$ distance-1 nodes. The latter nodes are at least distance $D – 1$ to $X$. So, the average distance from $X$ to the two set of nodes (neighbors of $X$ and $Y$) is at least $D/2$. This process can be repeated for distance-2, distance-3, … nodes, until done. ■
Some Practical Implications

$D$ and $\Delta$ are important network parameters.

Can’t judge a network merely on the basis of its aggregate bandwidth $Bw$.

Consider a 100-link network, with $Bw = 100b$.

Probability of being able to establish an $i$th random routing path of length $\Delta$ in the network is

$$p_i = \frac{(C-(i-1)\Delta)}{\Delta} / \frac{C}{\Delta}$$
Conclusions and Future Work

Calculating average distance avoidable in many cases

Ratio of diameter to average internode distance is:
- Unbounded in worst-case (impractical extremes)
- Between 1 and 2 in symmetric networks
- Fairly small in other practical cases
- Very close to 1 for trees

Future work and practical impact

- Tighten the bounds for special classes of networks
- Study pertinent bounds for Cayley graphs
- Simulate in detail effects of $D$ and $\Delta$
- Derive exact $\Delta$ for more networks
- Routing-based $D$ and $\Delta$
Questions or Comments?

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Back-Up Slides
Effect of $\Delta$ in Establishing Routing Paths

Probability of being able to establish an $i$th random routing path of length $\Delta$ in a 100-link network

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Routing with Wormhole Switching

Average internode distance $\Delta$ is an indicator of performance
$\Delta$ is closely related to the diameter $D$

For symmetric nets: $D/2 \leq \Delta \leq D$

Short worms: hop distance clearly dictates the message latency

Long worms: latency is insensitive to hop distance, but tied up links and waste due to dropped or deadlocked messages rise with hop distance