A Nonlinear Small Gain Theorem for the Analysis of Control Systems with Saturation

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Abstract—A nonlinear small gain theorem is presented that provides a formalism for analyzing the behavior of certain control systems that contain or utilize saturation. The theorem is used to show that an iterative procedure can be derived for controlling systems in a general nonlinear, feedforward form. This result, in turn, is applied to the control of: 1) linear systems (stable and unstable) with inputs subject to magnitude and rate saturation and time delays; 2) the cascade of globally asymptotically stable nonlinear systems with certain linear systems (those that are stabilizable, right invertible, and such that all of their invariant zeros have nonpositive real part); 3) the inverted pendulum on a cart; and 4) the planar vertical takeoff and landing aircraft.

I. INTRODUCTION

In this paper we develop a tool for analyzing the dynamical behavior of certain interconnected nonlinear systems, and we use this tool to propose feedback strategies for a class of nonlinear control systems. The analysis tool is a small gain theorem for interconnections where, for both subsystems in the interconnection, the asymptotic behavior of the input gives information about the asymptotic behavior of the output. In applying this analysis tool to various nonlinear control problems, we will be led to the control structure exploited in [30] and [28] for controlling linear systems with bounded inputs. We will see that these structures can be applied to a variety of nonlinear control problems, generalizing some of the results in [31].

Our “asymptotic” small gain theorem is, then, another tool for guiding nonlinear control design. The use of the small gain theorem for control design dates back to the 1960’s (see [35], [36], and [20], for example). The initial small gain theorem involved finite (linear) gains from a norm of the input to a norm of the output (see [4] for a summary). More recently, a small gain theorem involving monotone gains was developed in [15], following closely related ideas in [19]. In [9], this nonlinear small gain idea was applied to the interconnection of nonlinear systems that are input-to-state stable in the sense of Sontag (see [22]), a property expressed in terms of monotone gains. A general stability result was obtained and was subsequently used to solve several open robust nonlinear stabilization problems. In the present paper, we present a small gain theorem that uses a property similar to Sontag’s input-to-state stability property. However, it focuses primarily on the asymptotic behavior of the output with respect to the asymptotic behavior of the input. The theorem is especially concerned with the situation where some of the driving inputs are limited by saturation. As a special case, it will contain results on cascade interconnections (cf. [23] and [21]).

Even though we are predominately interested in asymptotic stability in the sense of Lyapunov, we do not employ Lyapunov functions explicitly. For the problems that we are considering and the controls that we are using, an input-output point of view seems more natural. However connections between input-output results and Lyapunov results are continually being made (see [7], [34], and [13], for example). Further, a Lyapunov approach for problems closely related to those we consider is being developed in [17]. This approach generates control laws that are more suitable for Lyapunov analysis. Ultimately, the emerging nonlinear input–output analysis tools should serve to complement Lyapunov-based tools, both helping to further equip the nonlinear control designer.

The remainder of the paper is organized as follows: In Section II we discuss some preliminary definitions and notation. In Section III-A we present a summary, in novel terms, of the results in [9] on the interconnection of nonlinear systems satisfying Sontag’s input-to-state stability property (cf. [3]). This section will serve to motivate the results in Section III-B on “saturated” interconnections where asymptotic input–output properties are introduced. In Section IV-A we review one of the main results in [14] which serves as the key building block for our control laws. It is used in Sections IV-B and IV-C, together with our small gain theorem, to produce a stabilization algorithm for nonlinear "feedforward" systems. Linear systems with input saturation is a special subclass of feedforward systems. So, in Section IV-D, we discuss the stabilization, by state or output feedback, of linear systems (stable and unstable) with inputs subject to input magnitude and rate saturation and time delays. In Section IV-E, we contribute a new result to the growing body of work on stabilization of nonlinear systems cascaded with linear systems (see [27] and [18], for example). Finally, in Section IV-F, we illustrate our control algorithms on two examples of nonlinear feedforward form systems: the inverted pendulum on a cart and the planar vertical takeoff and landing (PVTOL) aircraft (see [6]) with input corruption.

II. PRELIMINARIES

Throughout the paper we will use the vector ∞-norm, denoted ||·||, i.e., for u ∈ ℝ^n we define ||u|| := max_{i=1,...,n} |u_i|.
We will frequently make reference to the nonlinear system
\[ \dot{x} = f(x, u), \quad y = h(x, u) \]  
(1)
where \( f: \mathcal{X} \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) and \( h: \mathcal{X} \times \mathbb{R}^m \rightarrow \mathbb{R}^p \) are locally Lipschitz and \( \mathcal{X} \) is an open neighborhood of the origin\(^1\) and the following definitions.

1) A trajectory of (1), which depends on \( x(0) \) and \( u \), is said to:
   a) exist on \([0, T]\) if it is contained in \( \mathcal{X} \) for all \( t \in [0, T] \);
   b) be bounded on \([0, T]\) if it is contained in a compact subset of \( \mathcal{X} \) for all \( t \in [0, T] \).

2) The origin of the system
   \[ \dot{x} = f(x, 0) \]  
(2)
is said to be:
   a) stable if, for each \( \epsilon > 0, \exists \delta > 0 \) such that \( |x(0)| \leq \delta \) implies \( |x(t)| \leq \epsilon \) for all \( t > 0 \);
   b) locally asymptotically stable (LAS) if it is stable and there exists \( c > 0 \) such that \( |x(0)| \leq c \) implies \( x(t) \rightarrow 0 \) as \( t \to \infty \);
   c) locally exponentially stable (LES) if there exists strictly positive real numbers \( c, \mu \) such that \( |x(0)| \leq c \) implies \( |x(t)| \leq k|x(0)| \exp(-\mu t) \) for all \( t \geq 0 \).

3) Nonlinear system (1) is said to be:
   a) zero-input stable if the origin of (2) is stable;
   b) zero-input LAS (LES) if the origin of (2) is LAS (LES);
   c) zero-input LAS (LES) with basin of attraction \( \mathcal{A} \) if the origin of (2) is LAS (LES) and \( x(0) \in \mathcal{A} \) implies \( x(t) \rightarrow 0 \) as \( t \to \infty \) (It is this type of asymptotic stability that guarantees the existence of a Lyapunov function which is positive definite in \( \mathcal{A} \), proper w.r.t. \( \mathcal{A} \), and has a negative definite derivative along the trajectories of (2) which originate from \( \mathcal{A} \); see [11, Sec. 5]);
   d) zero-input globally asymptotically stable (GAS) if it is zero-input LAS with basin of attraction \( \mathcal{A} = \mathcal{X} \).

For the analysis portion of this paper, we will address various types of stability for the interconnection of nonlinear subsystems of the form (1) as in Fig. 1. In particular we will consider the system
\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, (y_2, d_1)), \quad y_1 = h_1(x_1, (y_2, d_1)) \\
\dot{x}_2 &= f_2(x_2, (y_1, d_2)), \quad y_2 = h_2(x_2, (y_1, d_2)).
\end{align*} \]  
(3)

(In the above equations, to avoid cumbersome notation, we have used \( u_1, u_2 \) with \( u_1 \in \mathbb{R}^{m_1}, u_2 \in \mathbb{R}^{m_2} \) to denote the vector \( u_1^T, u_2^T \in \mathbb{R}^{m_1 + m_2} \). We will use this throughout the paper.) Using the notation from the (1), the input to each subsystem is partitioned as \( u^{(1)} = (u_1^{(1)}, u_2^{(1)}) \), where \( u_1^{(1)} =
\]

\(^1\)On a first reading, one can consider that \( \mathcal{X} = \mathbb{R}^n \). We will need the more general setting when we consider the inverted pendulum, where we will have a controlled system which is not defined everywhere.

\(^2\)With the added property that the function is strictly increasing, such functions are often called class-K functions in the literature. We are enlarging that class to include functions like the identically zero function and the function \( \gamma(s) = \min \{ s, 1 \} \) for \( s \geq 0 \), for example.
Finally, for a measurable function $u : [0, T) \to \mathbb{R}^m$ and for any $t \in [0, T)$, $u_t$ is a function on $[0, \infty)$ defined by

$$u_t(\tau) = \begin{cases} u(\tau), & \text{if } \tau \in [0, t] \\ 0, & \text{otherwise.} \end{cases}$$

### III. Analysis Tools

#### A. Standard Interconnections

In this section we present some results on the interconnection of subsystems with outputs whose $L_\infty$-norm and asymptotic $L_\infty$-norm satisfy certain bounds. The results are mostly a combination of what can be found in [15], [29], [9], and [3]. One of the purposes of this section is to provide context for the new results given in Section III-B.

Let $I$ be a strictly positive integer satisfying $1 \leq m$, let $(u_1, \ldots, u_I)$ be a partition of $u \in \mathbb{R}^m$, let $X \subseteq X$, let $A := (A_1, \ldots, A_I)$ be an $I$-tuple of strictly positive real numbers (or possibly $\infty$), and let $\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_I)$ be an $I + 1$-tuple of gain functions.

**Definition 3.1:** The output function $h(x, u)$ for (1) is said to satisfy an $a-L_\infty$ stability bound from $X$ with gain $\gamma$ and restriction $A$ if, for each $x_0 \in X$ and each measurable $u$ satisfying $\|u\|_{L_\infty} < \Delta_i$, the solution of (1) with $x(0) = x_0$ exists for all $t \in [0, \infty)$ and satisfies

$$\|h(x, u)\|_{L_\infty} \leq \max_{i = 1, \ldots, I} \{\gamma_i(x_0)\}, \quad \max_{i = 1, \ldots, I} \gamma_i(\|u\|_{L_\infty})$$

$$\|h(x, u)\| \leq \max_{i = 1, \ldots, I} \gamma_i(\|u\|_{L_\infty}).$$

(5)

The function $\gamma_i$ is referred to as the channel $i$ gain and $\Delta_i$ as the channel $i$ restriction.

It is very natural to work with the above property when one is studying asymptotic stability in the sense of Lyapunov. Indeed, if the state satisfies an $a-L_\infty$ stability bound from $X$, an open neighborhood of the origin, then when $u \equiv 0$ and $x(0) \in X$, $\|x\|_{L_\infty} \leq \gamma_0(\|x(0)\|)$ and $\|x\|_{L_\infty} = 0$, i.e., the origin is LAS with basin of attraction containing $X$. (For more general output functions establishing asymptotic stability requires extra “observability” conditions relating $\|x\|_{L_\infty}$ to $\|y\|_{L_\infty}$ and $\|z\|_{L_\infty}$ to $\|u\|_{L_\infty}$. We will not need such conditions for our control applications so we do not pursue this direction.)

It is instructive to compare the property above in the case where $h(x, u) = x$, $X = X = \mathbb{R}^n$, $I = 1$, and $\Delta_1 = \infty$ to the input-to-state stability property given in [22]. There, a system is said to be input-to-state stable if there exists a gain function $\gamma$ and a function $\beta$ of class-$KL$ such that for each $x_0 \in \mathbb{R}^n$ and each essentially bounded $u$, the solution exists for all $t \geq 0$ and satisfies

$$\|x(t)\| \leq \max \{\beta(\|x_0\|, t), \gamma(\|u\|_{L_\infty})\} \quad \forall t \geq 0.$$  

(6)

Note that by setting $t = 0$ in (6), the first inequality in (5) is obtained with $\gamma_0(s) = \beta(s, 0)$ and $\gamma_1(s) = \gamma(s)$. Moreover, using time invariance and the bound on $\|x\|_{L_\infty}$, the second inequality of (5) is obtained with $\gamma_1(s) = \gamma(s)$. In fact, the work in [25] indicates that the state of (1) satisfies an $a-L_\infty$ stability bound if and only if the system is input-to-state stable.

On the other hand, it is not true that if the state satisfies an $a-L_\infty$ stability bound with gain $(\gamma_0, \gamma_1)$ then the bound (6) holds with $\gamma = \gamma_1$. For example, the state of the system

$$\dot{x} = \frac{1}{1 + x^2}$$

(7)

satisfies an $a-L_\infty$ stability bound with gain $\gamma_1 \equiv 0$. Yet, the system does not satisfy (6) with $\gamma \equiv 0$ for any $\beta$ of class-$KL$ since the convergence rate to zero decreases as $\|u\|_{L_\infty}$ increases.

For the case where $h(x, u) \neq x$, other input-output stability definitions are given in [22] and [9]. In [9], particular attention is given to decoupling the property that the solution exists on $[0, \infty)$ from the (truncated) $L_\infty$ input-output behavior. Extra observability conditions are imposed to guarantee boundedness of the state. The following lemma describes to what extent observability is already captured in the definition of an $a-L_\infty$ stability bound.

**Lemma 3.1:** Suppose the output of (1) satisfies an $a-L_\infty$ stability bound with gain $\gamma$ and restriction $\Delta$. Let $T > 0$ be finite. If $u$ is defined on $[0, T)$ and for each $\iota \in \{1, \ldots, m\}$ we have $\sup_{t \in [0, T)} \|u_\iota(t)\| < \Delta_\iota$, then the state is bounded on $[0, T)$.

**Proof:** Let $v$ be a function defined on $[0, \infty)$ satisfying $v(t) = u(t)$ for $t \in [0, T)$ and $v(t) = 0$ otherwise. Then $v$ satisfies the restriction $\Delta$ and so the solution of $\dot{x} = f(x, v)$ is zero-input and thus bounded on $[0, T)$. But, from causality, the solution $x(t)$ agrees with $x(t)$ on $(0, T)$, yielding the conclusion of the lemma.

We note that an $a-L_\infty$ stability bound on the state is guaranteed in the following situations.

**Lemma 3.2:** If the origin of (1) is zero-input LAS, then there exists an open neighborhood $X$ of the origin, a strictly positive real number $\Delta$ and a gain function $\gamma$ such that the state of (1) satisfies an $a-L_\infty$ stability bound from $X$ with gain $\gamma$ and restriction $\Delta$. If the origin of (1) is zero-input LES, then $\gamma$ can be taken to be a linear function.

**Proof:** See [10, Th. 4.10] (cf. [5, Sec. 56]).

**Lemma 3.3:** Suppose there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, globally invertible gain functions $\alpha_1$ and $\alpha_2$, a gain function $\gamma$, and strictly positive real numbers (or possibly $\infty$) $\delta_0$ and $\delta_\varepsilon$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

(8)

and

$$\gamma(|u|) < |x| < \delta_\varepsilon, \quad |u| < \delta_u \implies \frac{\partial V}{\partial x} f(x, u) < 0.$$  

(9)

Let $r$ and $\Delta$ be strictly positive real numbers (or possibly $\infty$) satisfying

$$\alpha_1^{-1} \circ \alpha_2(r) \leq \delta_\varepsilon, \quad \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\Delta) \leq \delta_\varepsilon, \quad \Delta \leq \delta_u.$$  

(10)

Then the state of $\dot{x} = f(x, u)$ satisfies an $a-L_\infty$ stability bound from the set $\{x \in \mathbb{R}^n : |x| < r\}$ with gain $(\alpha_1^{-1} \circ \alpha_2, \alpha_1^{-1} \circ \alpha_2 \circ \gamma)$ and restriction $\Delta$. 


Proof: Suppose $x_0 \in \{x \in \mathbb{R}^n : |x| < \tau \}$ and $\|u\|_\infty < \Delta$. From (10), (8), and the assumption on $u$ it follows that $\gamma(\|u\|_\infty) < \delta_2$ and $\|u\|_\infty < \delta_0$. Also note from (10) that $\alpha_1^{-1} \circ \alpha_2(\max \{|x_0|, \gamma(\|u\|_\infty)\}) < \delta_2$. Since $\gamma(\|u\|_\infty) < |x| < \delta_2$ and $\|u\|_\infty < \delta_0$ imply that $V < 0$ (almost everywhere in time), it follows that $x$ is defined on $[0, \infty)$ and $\|x\|_\infty \leq \alpha_1^{-1} \circ \alpha_2(\max \{|x_0|, \gamma(\|u\|_\infty)\})$, giving the first inequality of (5) with $\gamma_0 = \alpha_1^{-1} \circ \alpha_2$ and $\gamma_1 = \alpha_1^{-1} \circ \alpha_2 \circ \gamma$. This property for $V$ also tells us that $x$ converges to a closed ball of radius $\alpha_1^{-1} \circ \alpha_2(\gamma(\|u\|_\infty))$. Using time invariance, the second inequality of (5) holds with $\gamma_1$ given above.

In this section, we are particularly interested in when the Lipschitz well-posed interconnection of systems with outputs satisfying $a$-$L_\infty$ stability bounds is such that the composite output satisfies an $a$-$L_\infty$ stability bound with respect to external disturbances. We will suppose that the output of the $i$th ($i = 1, 2$) subsystem in (3) satisfies an $a$-$L_\infty$ stability bound from $X_i$ with gain $(\gamma_i(0), \gamma_i(1), \gamma_i(2))$ and restriction $(\Delta_i(1), \Delta_i(2))$, and we will impose a condition on the channel one gains, $\gamma_i(1)$ and $\gamma_i(2)$, to guarantee that the composite output also satisfies an $a$-$L_\infty$ stability bound. The condition will be that the composition of the gain functions is a simple contraction. The composition of two gain functions, $\gamma_1$ and $\gamma_2$, is a simple contraction if $\gamma_1(\gamma_2(s)) < s$ (equivalently, $\gamma_2(\gamma_1(s)) < s$) for all $s > 0$. The gains and restrictions of the $a$-$L_\infty$ stability bound for the composite output will be defined in terms of the gains and restrictions of the subsystems. We make the definition now to make the statement of the theorem more concise.

1) $\bar{X}_1$ and $\bar{X}_2$ are sets defined as

$$\bar{X}_1 := \{x \in X_1 : \gamma_1^{(1)}(|x|) < \Delta_1^{(2)}, \gamma_1^{(2)} \circ \gamma_0^{(1)}(|x|) < \Delta_1^{(1)}\} \cap X_1$$
$$\bar{X}_2 := \{x \in X_2 : \gamma_2^{(1)}(|x|) < \Delta_2^{(1)}, \gamma_1^{(1)} \circ \gamma_0^{(2)}(|x|) < \Delta_2^{(2)}\} \cap X_2.$$ (11)

2) $\Delta_1$ and $\Delta_2$ are strictly positive real numbers (or possibly infinite) satisfying

$$\Delta_1 \leq \Delta_1^{(1)}, \quad \Delta_2 \leq \Delta_2^{(2)}.$$ (12)

3) $\gamma_0$, $\gamma_1$, and $\gamma_2$ are gain functions defined as $\gamma_0(s) = \max \{\gamma_0^{(1)}(s), \gamma_1^{(1)} \circ \gamma_0^{(2)}(s), \gamma_0^{(2)}(s), \gamma_2^{(1)} \circ \gamma_0^{(1)}(s)\}$ $\gamma_1(s) = \max \{\gamma_2^{(1)}(s), \gamma_1^{(2)} \circ \gamma_2^{(1)}(s)\}$ $\gamma_2(s) = \max \{\gamma_1^{(1)} \circ \gamma_2^{(2)}(s), \gamma_2^{(2)}(s)\}.$ (13)

Notice that if $\Delta_1^{(1)} = \infty$ and $\Delta_1^{(2)} = \infty$, then $\bar{X}_1 \times \bar{X}_2 = X_1 \times X_2$. This is a simple contraction with gain $(\gamma_1^{(1)}, \gamma_1^{(2)}, \gamma_2^{(2)})$ and restriction $(\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_2^{(2)})$. For the particular case where either $\Delta_1^{(1)}$ or $\Delta_1^{(2)}$ is finite, we will use the following mild technical assumption.

Assumption 1: $X_1$ and $X_2$ are path connected neighborhoods of the origin, and when the initial condition is the origin and $d \equiv 0$ the solutions are defined on $[0, \infty)$ and the composite output is identically zero.

Theorem 1: Suppose that the output of the $i$th ($i = 1, 2$) subsystem in (3) satisfies an $a$-$L_\infty$ stability bound from $X_i$ with gain $(\gamma_i^{(0)}, \gamma_i^{(1)}, \gamma_i^{(2)})$ and restriction $(\Delta_i^{(1)}, \Delta_i^{(2)})$. When $\Delta_i^{(1)}$ or $\Delta_i^{(2)}$ is finite also suppose that Assumption 1 holds. If the channel one gains form a simple contraction then, using the definitions in (11)–(13), the composite output $(\bar{y}_1, \bar{y}_2)$ for (3) satisfies an $a$-$L_\infty$ stability bound from $\bar{X}_1 \times \bar{X}_2$ with gain $(\gamma_0, \gamma_1, \gamma_2)$ and restriction $(\Delta_1, \Delta_2)$.

Proof: The proof for the case where either $\Delta_1^{(1)}$ or $\Delta_1^{(2)}$ is finite is more complicated and is thus deferred to the Appendix. If $\Delta_1^{(1)} = \Delta_1^{(2)} = \infty$, then the system is a simple contraction with gain $(\gamma_0^{(2)}, \gamma_1^{(2)}, \gamma_2^{(2)})$ and restriction $(\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_2^{(2)})$. The composite output $(\bar{y}_1, \bar{y}_2)$ for (3) satisfies an $a$-$L_\infty$ stability bound from $\bar{X}_1 \times \bar{X}_2$ with gain $(\gamma_0, \gamma_1, \gamma_2)$ and restriction $(\Delta_1, \Delta_2)$.

Combining, we get

$$||y_1||_\infty \leq \max \{\gamma_0^{(1)}(|x_1|), \gamma_1^{(1)} \circ \gamma_0^{(2)}(|x_2|), \gamma_0^{(2)}(|x_1|)\}$$
$$||y_2||_\infty \leq \max \{\gamma_1^{(2)}(|x_1|)\}$$ (14)

and

$$||y_2||_\infty \leq \max \{\gamma_2^{(2)}(|x_2|), \gamma_1^{(1)} \circ \gamma_0^{(1)}(|x_1|)\} = \max \{\gamma_2^{(2)}(|x_2|), \gamma_0^{(2)}(|x_2|)\}$$ (15)

But, since the composition of $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ is a simple contraction, it follows that

$$||y_1||_\infty \leq \max \{\gamma_0^{(1)}(|x_1|), \gamma_1^{(1)} \circ \gamma_0^{(2)}(|x_2|), \gamma_0^{(2)}(|x_1|)\}$$
$$||y_2||_\infty \leq \max \{\gamma_1^{(2)}(|x_2|), \gamma_1^{(1)} \circ \gamma_0^{(1)}(|x_1|), \gamma_1^{(1)} \circ \gamma_0^{(1)}(|x_1|)\}$$ (17)

(If this were not the case, then the term involving the composition of $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ would be the maximum in the list and...
we would have an inequality of the form \( s \leq \gamma_1^{(1)} \circ \gamma_1^{(2)}(s) \) which contradicts that the composition is a simple contraction.

Now, since the right-hand sides are independent of \( t \), we have that \( \sup_{t \in [0,T]} |y(t)| < \infty \) and \( \sup_{t \in [0,T]} |y_2(t)| < \infty \). Using Lemma 3.1 we have that if \( T \) is finite then the state is also bounded on \([0, T]\). This contradicts \([0, T]\) being the maximal interval of definition and thus we conclude that \( T = \infty \). Letting \( t \to \infty \) on the left-hand side of (17) and using that \( \|y_1(t), y_2(t)\| = \max \{\|y_1\|_\infty, \|y_2\|_\infty\} \) gives us that the composite output function satisfies the first inequality of the \( a-L_\infty \) stability bound from \( \tilde{X}_1 \times \tilde{X}_2 \) with gain \( (\gamma_0, \gamma_1, \gamma_2) \) as defined in (13) and with restriction \((\Delta_1, \Delta_2)\) as defined in (12).

We also now have that \( \|y_1\|_a \) and \( \|y_2\|_a \) are well-defined so that we can also use

\[
\|y_1\|_a \leq \max \{\gamma_1^{(1)}(\|y_2\|_a), \gamma_1^{(2)}(\|d_2\|_a)\}
\]

\[
\|y_2\|_a \leq \max \{\gamma_2^{(1)}(\|y_1\|_a), \gamma_2^{(2)}(\|d_2\|_a)\}.
\] (18)

Combining these inequalities and again using that the composition of \( \gamma_1^{(2)} \) and \( \gamma_1^{(2)} \) is a simple contraction, the composite output function also satisfies the second inequality of the \( a-L_\infty \) stability bound with the same gain and restrictions.

\[\square\]

B. "Saturated" Interconnections

The main analytical tool for the control results in this paper is a small gain theorem for interconnections where the restriction for the driving inputs is only on \( \|u\|_a \) rather than on \( \|u\|_\infty \). In fact, \( L_\infty \) bounds are not used at all. Instead, we will be particularly interested in interconnections where one subsystem exhibits asymptotically small outputs only for asymptotically small inputs, while the other subsystem exhibits asymptotically small outputs regardless of the size of the inputs. Again, let \( l \) be a strictly positive integer satisfying \( l \leq m \), let \((u_1, \ldots, u_l)\) be a partition of \( u \in \mathbb{R}^m \), let \( X \subset \mathcal{X}_0 \), let \( \Delta := (\Delta_1, \ldots, \Delta_l) \) be an \( l \)-tuple of nonnegative real numbers (or possibly \( \infty \)), and let \( \gamma := (\gamma_1, \ldots, \gamma_l) \) be an \( l \)-tuple of gain functions.

**Definition 3.2**: The output function \( h(x, u) \) for (1) is said to satisfy an asymptotic bound from \( X \) with gain \( \gamma \) and restriction \( \Delta \) if, for each \( x_0 \in X \) and each locally essentially bounded \( u \) satisfying \( \|u\|_a \leq \Delta_i \), the solution of (1) with \( x(0) = x_0 \) exists for all \( t \in [0, \infty) \) and satisfies

\[\|h(x, u)\|_a \leq \max_{i=1, \ldots, m} \gamma_i(\|u_i\|_a).\] (19)

An obvious difference between an asymptotic bound and an \( a-L_\infty \) stability bound is that \( L_\infty \) norm bounds are not used and, accordingly, there is no gain from the norm of the initial condition to some measure of the output. More subtly, notice that the restrictions are expressed in terms of simple inequalities for an asymptotic bound rather than in terms of strict inequalities. First, this indicates that it makes sense to talk about the situation where \( \Delta = 0 \). Second, it indicates that when \( \Delta_1 = \infty \) we only need to establish that \( u_1 \) is locally essentially bounded, rather than essentially bounded, before we use (19). Third, it indicates that an \( a-L_\infty \) stability bound from \( X \) with gain \( \gamma_1 \) and restriction \( \Delta_1 = \infty \) does not necessarily imply an asymptotic bound from \( X \) with gain \( \gamma_1 \) and restriction \( \Delta_1 = \infty \). For example, consider the system given in (7) which satisfies an \( a-L_\infty \) stability bound from \( \mathbb{R}^m \) with gain \( \gamma_1 \equiv \infty \) and restriction \( \Delta_1 = \infty \). The system does not satisfy an asymptotic bound with this same gain and restriction since there is no guarantee that \( x \) converges to zero when \( u \) diverges. On the other hand, the following result holds.

**Lemma 3.4**: Suppose an output function satisfies an \( a-L_\infty \) stability bound from \( X \) with gain \((\gamma_0, \gamma_1, \gamma_2)\) and restriction \( \Delta_1 \).

If (1) \( \Delta_1 = \infty \) or (2) \( \Delta_1 < \infty \), no finite escape times are possible and \( X = \mathcal{X}_0 \), then the output function satisfies an asymptotic bound from \( X \) with gain \( \gamma_1 \) and restriction \( \Delta_1 \) for any \( \Delta_1 < \Delta_1 \).

If the state of (1) satisfies an asymptotic bound from an open neighborhood of the origin, this is not enough to guarantee that the system is zero-input LAS. Namely, stability is not guaranteed even though convergence to the origin is. Instead, we will typically check the zero-input LAS property from the Jacobian linearization or Theorem 1. Then, we will get information about the domain of attraction using the set \( X \) in the characterization of an asymptotic bound.

As in the previous section, we are interested in when the Lipschitz well-paced interconnection of systems with outputs satisfying an asymptotic bound is such that the composite output satisfies an asymptotic bound with respect to external disturbances. Once again we will require that the channel one gains form a simple contraction. This, and an additional compatibility condition, will give us that the outputs of the interconnection (3) satisfy an asymptotic bound from \( X_1 \times X_2 \) with gain \((\gamma_1, \gamma_2)\) given in (13) and restriction \((\Delta_1, \Delta_2)\), where \( \Delta_1 \) and \( \Delta_2 \) are nonnegative real numbers satisfying

\[\Delta_1: \Delta_1 = \Delta_2^{(1)} \]
\[\Delta_2: \Delta_2 \leq \Delta_2^{(2)}, \quad \max \{\gamma_2^{(1)}(\infty), \gamma_2^{(2)}(\Delta_2)\} \leq \Delta_2^{(1)}.\] (20)

The additional compatibility condition will be that \( \gamma_2^{(2)}(\infty) \leq \Delta_1^{(1)} \) so that such a \( \Delta_2 \) exists. Because we do not have \( L_\infty \) bounds to keep track of the size of the outputs on finite time intervals, we will simply assume that there are no finite escape times in Theorem 2 below. In the applications that follow, we will be able to check that no finite escape times are possible with the following lemma.

**Lemma 3.5**: Consider a Lipschitz well-paced system of the form (3) where the output of the \( i \)th subsystem \((i = 1, 2)\) satisfies an asymptotic bound from \( X_i \), Suppose \( T > 0 \) is finite, \( d \) is locally essentially bounded, and \( x_0 = x_1 \times X_2 \). If either one of the outputs is (essentially) bounded on \([0, T]\), then the composite state is bounded on \([0, T]\).

**Proof**: Suppose, without loss of generality, that \( y_1 \) is (essentially) bounded on \([0, T]\), and let \( v_1, v_2 \) be signals defined on \([0, \infty)\) satisfying \( v_1(t) = y_1(t) \) for \( t \in [0, T) \), \( v_1(t) = 0 \) otherwise and \( v_2(t) = d_2(t) \) for \( t \in [0, T) \), \( v_2(t) = 0 \) otherwise. Note that \( v_1 \) and \( v_2 \) are essentially bounded and \( \|v_1\|_a = \|.v_2\|_a = 0 \). Then, from the asymptotic bound on the output, the solution of \( \dot{x}_2 = f_2(x_2, v_1, v_2) \),
Consider a general class of saturation functions as characterized in the following definition, adapted from [14].

Definition 4.1: A function \( \sigma : \mathbb{R}^n \to \mathbb{R}^m \) is said to be a saturation function if it is differentiable at the origin, decentralized, i.e., the \( i \)-th component of \( \sigma(u) \) depends only on the \( i \)-th component of \( u \), for each \( i \in \{ 1, \cdots, m \} \), we have
\[
\sigma_i(s) > 0 \quad \text{for all} \quad s \neq 0,
\]
and there exist \( K > 0, b > 0 \) such that for all \( s, t \in \mathbb{R} \)
\[
1) \quad |\sigma_i(t + s) - \sigma_i(t)| \leq \min \{ K|s|, b \};
\]
\[
2) \quad |t| |\sigma_i(t + s) - \sigma_i(t)| \leq K|s|;
\]
\[
3) \quad |\sigma_i(t) - t| \leq Kt\sigma_i(t).
\]

Remark 4.1: It follows from the definition that \( \frac{d\sigma_i}{ds}|_{s=0} = \sigma_i'(0) = I_{m \times m} \). This class of functions includes the standard saturation function \( \text{sat}_i(s) = \text{sgn}(s) \min \{ |s|, 1 \} \). However, to simplify later developments, we have taken the class to be slightly less general than the class considered in [14].

The first results we will need for our applications come from [14].

Lemma 4.1: Suppose the pair \((A, B)\) is stabilizable and there exists \( P > 0 \) such that \( AT^TP + PA \leq 0 \), i.e., \( A \) is critically stable, but not necessarily Hurwitz. Let \( \sigma \) be a saturation function. Then the system
\[
\dot{x} = Ax + B\sigma(-B^TPx + d_1) + d_2
\]
is zero-input LES and its state satisfies an asymptotic bound from \( \mathbb{R}^n \) with linear gain and nonzero restriction with respect to \( d = (d_1, d_2) \).

Proof: See [14].

Under the conditions of the lemma, there may be other feedbacks \( u = Fx \) for the system
\[
\dot{x} = AX + B\sigma(u + d_1) + d_2
\]
which give the same result. For example, let \( z = Tx \) represent a coordinate change giving
\[
\dot{z} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \sigma(u + d_1) + \overline{d}_2
\]
where \( \overline{A}_{22} \) is Hurwitz. Necessarily, we must have that \((\overline{A}_{11}, \overline{B}_1)\) is stabilizable and there exists \( \overline{P} > 0 \) such that \( \overline{A}_{11}^T\overline{P} + \overline{P}\overline{A}_{11} \leq 0 \). So, using Lemma 4.1 for the \( z_1 \) subsystem and Theorem 2 for the connection with the \( z_2 \) subsystem, we have that the feedback \( u = -\overline{B}_1^T\overline{P}[T_{11} \ T_{12}]x \) also gives the conclusion of Lemma 4.1 for (24). This discussion motivates the following definitions. When \((A, B)\) is stabilizable and there exists \( P > 0 \) such that \( A^TP + PA \leq 0 \), \((A, B)\) is said to admit a good saturated linear controller. If \( F \) is such that \( A + BF \) is Hurwitz and the state of (24) with \( u = Fx \) satisfies an asymptotic bound from \( \mathbb{R}^n \) with linear gain and nonzero restriction with respect to \( d = (d_1, d_2) \), then \( F \) is said to be a good saturated linear controller for \((A, B)\).

Finally, we remind the reader of the following result, noted in [28] in a more general form.
Lemma 4.2: Let σ be any Lipschitz function. If the state $x \in \mathbb{R}^n$ of the system $\dot{x} = Ax + Bσ(Fx + d_1) + d_2$ satisfies an asymptotic bound from $\mathbb{R}^n$ with gain $N \cdot \text{Id}$ and restriction $\Delta$ with respect to $d = (d_1, d_2)$, then, for each $\lambda > 0$, the state of the system

$$\dot{x} = Ax + B\sigma \left( \frac{Fx + d_1}{\lambda} \right) + d_2$$  \hspace{1cm} (26)

satisfies an asymptotic bound from $\mathbb{R}^n$ with gain $N \cdot \text{Id}$ and restriction $\lambda \Delta$ with respect to $d = (d_1, d_2)$.

**Proof:** Define the new coordinate $z = x/\lambda$ so that (26) becomes

$$\dot{z} = Az + B\sigma \left( \frac{Fz + d_1}{\lambda} \right) + \frac{d_2}{\lambda}.$$  \hspace{1cm} (27)

By assumption, $\|d/\lambda\|_a \leq \Delta$ implies $\|z\|_a \leq N\|d/\lambda\|_a$. Then, since $z = \lambda x$, the conclusion follows. \hfill $\square$

**B. A General Stabilization Result**

In the following theorem we show that the stability produced by a good saturated linear controller is robust to a certain class of additive dynamic perturbations. The theorem also helps us design controllers for the class of so-called feedforward nonlinear systems (see the ensuing discussion and Theorems 4 and 5).

**Theorem 3:** Consider the locally Lipschitz control system

$$\dot{x}_1 = Ax_1 + Bu + g(x_2, u, d)$$

$$\dot{x}_2 = f(x_2, u, d)$$  \hspace{1cm} (28)

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Suppose

1) $(A, B)$ admits a good saturated linear controller;
2) with respect to $(u, d)$, the state of the $x_2$ subsystem satisfies an asymptotic bound from $X$ with linear gains and a nonzero channel one restriction;
3) $\lim_{(x_2, u, d) \to (0,0,0)} g(x_2, u, d) = 0$.

Let $\sigma$ be a saturation function, let $\Gamma, \Omega \in \mathbb{R}^{m \times m}$, and let $F$ be a good saturated linear controller for $(A, B)$. Then there exists a strictly positive real number $\lambda$ such that with the control

$$u = \lambda \sigma \left( \frac{Fx_1 + \Gamma v}{\lambda} \right) + \Omega v$$  \hspace{1cm} (29)

and with respect to $(v, d)$, the state of (28) satisfies an asymptotic bound from $\mathbb{R}^{n_1} \times X$ with linear gains and a nonzero channel one restriction.

**Proof:** We will use $k_\lambda(x_1, v)$ to denote the right-hand side of (29). Consider the interconnection

$$\dot{x}_1 = Ax_1 + Bk_\lambda(x_1, v) + y_2,$$

$$\dot{x}_2 = f(x_2, y_1, d) + y_2 = g(x_2, y_1, d).$$  \hspace{1cm} (30)

Since $v$ is assumed to be locally essentially bounded and $\sigma$ is a saturation function, $k_\lambda(x_1, v)$ is (essentially) bounded on the interval $[0, T)$ for any finite $T$. Thus, from Lemma 3.5 it follows that no finite escape times are possible.

We now make two claims. In the first claim we will use the value $b$ that characterizes the saturation function $\sigma$.

**Claim 1:** There exist strictly positive real numbers $\Delta^{(1)}_2$, $L^{(1)}_2$, and $L^{(1)}_1$, independent of $\lambda$, such that, defining $\gamma^{(1)}_1(s) = 2 \min \{ \lambda b, L^{(1)}_1 \cdot s \}$, $y_1$ satisfies an asymptotic bound with respect to $(y_2, v)$ from $\mathbb{R}^n$ with gain $(\gamma^{(1)}_1, L^{(1)}_2 \cdot \text{Id})$ and restriction $(\infty, \lambda \Delta^{(1)}_2)$.

**Claim 2:** Let $L^{(2)}_2$ be a strictly positive real number such that $L^{(2)}_1 \cdot \Delta^{(2)}_1 < 1$. Then there exist strictly positive real numbers $\Delta^{(2)}_1$ and $L^{(2)}_1$ and a nonnegative real number $\Delta^{(2)}_2$ such that, with respect to $(y_1, d)$, $y_2$ satisfies an asymptotic bound from $X$ with gain $(L^{(2)}_1 \cdot \text{Id}, L^{(2)}_2 \cdot \text{Id})$ and restriction $(\Delta^{(2)}_1, \Delta^{(2)}_2)$.

Supposing these claims to be true, let $\lambda > 0$ satisfy

$$\max \{ 2\lambda b, L^{(2)}_2 \Delta^{(1)}_2 \} \leq \Delta^{(2)}_1.$$  \hspace{1cm} (31)

Then, according to Theorem 2, the output $(y_1, y_2)$ satisfies an asymptotic bound with respect to $(v, d)$ from $\mathbb{R}^n \times X$ with linear gains (since $\gamma^{(2)}_1$ can be upper bounded by the linear gain function $2L^{(1)}_1 \cdot \text{Id}$ and a nonzero channel one restriction. With these asymptotic bounds on $y_1$ and $y_2$, the appropriate asymptotic bounds on the composite state follows from the second assumption of the theorem and Lemma 4.1.

**Proof of Claim 1:** By the assumption on $F$ and Lemma 4.2, there exist strictly positive real numbers $N$ and $\Delta$ such that the state $x_1$ satisfies an asymptotic bound from $\mathbb{R}^{n_1}$ with gain $N \cdot \text{Id}$ and restriction $\lambda \Delta$ with respect to $d = (y_2, v)$. From the properties of a saturation function, there exists $L > 0$ such that

$$|k_\lambda(x_1, v)| \leq \max \{ \min \{ 2\lambda b, L \cdot |x_1| \}, L |v| \}.$$  \hspace{1cm} (32)

Using the asymptotic bound on the state, we get that

$$\max \{ ||y_2||_a, ||v||_a \} \leq \lambda \Delta$$  \hspace{1cm} (33)

implies

$$||k_\lambda(x_1, v)||_a \leq \max \{ \min \{ 2\lambda b, L \max \{ N \cdot ||y_2||_a, N \cdot ||v||_a \} \}, L \cdot ||v||_a \}.$$  \hspace{1cm} (34)

Now, using the fact that for any nonnegative real numbers $a, b, c$

$$\min \{ a, \max \{ b, c \} \} \leq \max \{ \min \{ a, b \}, c \}$$  \hspace{1cm} (35)

(just consider the two cases $b \leq c$ and $b \geq c$), we get that (33) implies

$$||k(x_1, v)||_a \leq \max \{ 2\lambda b, L \cdot N \cdot ||y_2||_a \}, L \cdot (N + 1) \cdot ||v||_a \}.$$  \hspace{1cm} (36)

So far we have the result we want except that the restriction on $y_2$ is currently $\lambda \Delta$ rather than $\infty$. To relax the restriction on $y_2$, let us assume, without loss of generality, that $N \geq 2\lambda b / (\lambda \Delta)$. In this case, when $||y_2||_a \geq \lambda \Delta$ it follows that

$$\min \{ 2\lambda b, L \cdot N \cdot ||y_2||_a \} = 2\lambda b.$$  \hspace{1cm} (37)

So it can be seen that the bound in (36) holds for $||y_2||_a \geq \lambda \Delta$ by applying $|| \cdot ||_a$ to (32).
Proof of Claim 2: From Assumption 3 of the lemma and the local Lipschitz property of \( g \), there exists a nonnegative real number \( L \), and for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that:

\[
|g(x_2, y_1, d)| \leq \max \{ |g(x_2, y_1)|, |Ld| \}.
\]

(37)

Let \((N_a(2)^t \cdot I_d, N_d(2)^t \cdot I_d)\) be the linear gains and \((\Delta_a(2)^t, \Delta_d(2)^t)\) be the restrictions indicated by Assumption 2 of the theorem. With \( L_1(1)^t \) a strictly positive real number satisfying \( L_1(2)^t, L_1(1)^t < 1 \), choose \( \epsilon = L_1(2)^t / \max \{1, N_a(2)^t\} \). This fixes \( \delta(\epsilon) > 0 \). Now pick \( \Delta_a(2)^t \) and \( \Delta_d(2)^t \) to satisfy

\[
\Delta_a(2)^t \leq \min \{ \Delta_a(2)^t, \delta / N_a(2)^t, \delta \}
\]

\[
\Delta_d(2)^t \leq \min \{ \Delta_d(2)^t, \delta / N_d(2)^t, \delta \}
\]

(38)

and define \( L_2(2)^t = \max \{ L, \epsilon N_d(2)^t \} \). Then the result follows from applying \( \| \cdot \|_a \) to (37) and using the asymptotic bound on the state \( x_2 \).

\( \square \)

Remark 4.2: From the proof of Claim 2 it should be clear that Assumption 3 of the lemma can be relaxed. Indeed, we only require that \( g(x, u, 0) \) be locally Lipschitz with a suitably small Lipschitz constant near the origin. This remark has implications for the robustness of our subsequent control algorithms. However, to keep the discussion simple, we will not take up this issue in detail.

We now will use the previous theorem to develop an iterative control design procedure for the class of so-called feedforward nonlinear systems (see Theorem 5). Consider the system

\[
\dot{x}_{i+1} = \begin{bmatrix} \dot{x}_a \vert \dot{x}_i \end{bmatrix} = f_i(x_i, u, d)
\]

(39)

where \( x_a \in \mathbb{R}^n, x_i \in X_i \subset \mathbb{R}^n, u \in \mathbb{R}^m \) and \( d \in \mathbb{R}^l \).

Note the cascade structure where \( x_a \) is the result of passing the output \( f_a(x_i, u, d) \) of the \( x_i \) subsystem through the critically stable linear system \((A_{x_i}, I)\). We will show that if the linearization of the composite system is stabilizable and we can appropriately stabilize (as specified in Definition 4.2 below) the \( x_i \) subsystem and its output \( f_a \), then we can stabilize the composite system in the same sense.

Definition 4.2: Let \( X \) be an open neighborhood of the origin in \( \mathbb{R}^n \) and consider the locally Lipschitz system

\[
\dot{x} = f(x, u, d)
\]

(40)

with \( x \in X, f: X \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n \). The set \( X \subseteq X \) and the function \( k: X \times \mathbb{R}^m \to \mathbb{R}^m \) satisfy the induction hypothesis for (40) if:

1) \( k \) is locally Lipschitz and differentiable at the origin;
2) the matrix

\[
\frac{\partial k(x, v)}{\partial v} \bigg|_{x=0,v=0}
\]

is invertible;
3) the matrix

\[
\frac{\partial f(x, k(x, v), u)}{\partial x} \bigg|_{x=0,v=0,d=0}
\]

(42)

is Hurwitz;
4) the state of the system

\[
\dot{x} = f(x, k(x, v), d)
\]

(43)

with input \((v, d)\) satisfies an asymptotic bound from \( X \) with linear gains and a nonzero channel one restriction.

In the following theorem, we will refer to the Jacobian linearization of (39) with \( u = k_i(x_i, v) \) and \( d = 0 \), i.e.,

\[
\begin{align*}
\dot{x}_a &= A_{x_i} x_a + f_a(x_i, k_i(x_i, v), 0) \\
\dot{x}_i &= f_i(x_i, k_i(x_i, v), 0).
\end{align*}
\]

(44)

Theorem 4: If the Jacobian linearization of (39) is stabilizable, if \( A_{x_i} \) is critically stable, and if the function \( k_i(x_i, v) \) and the set \( X_i \) satisfy the induction hypothesis for the \( x_i \) subsystem and its output \( k_i \) satisfy the induction hypothesis for \( (39) \) then there exists a function \( k_{i+1}(x_{i+1}, v) \) and a set \( X_{i+1} \) satisfying the induction hypothesis for (39). In particular, the above assumptions imply that the Jacobian linearization of (44) admits a good saturated linear controller \( F \). If \( \sigma \) is a saturation function and the matrices \( \Gamma, \Omega \in \mathbb{R}^{n \times m} \) are such that \( \Gamma + \Omega \) is invertible, then there exists a strictly positive real number \( \lambda \) such that the set \( X_{i+1} = \mathbb{R}^n \times X_i \) and the function

\[
k_{i+1}(x_{i+1}, v) = k_i \left( x_i, \lambda \sigma \left( \frac{f_{i+1}(x_{i+1} + \Gamma v)}{\lambda} + \Omega v \right) \right).
\]

(45)

satisfy the induction hypothesis for the \( x_{i+1} \) subsystem.

Proof: Since the Jacobian linearization of (39) is stabilizable, since the matrix in (41) is invertible, since the matrix in (42) is Hurwitz, since the matrix \( A_{x_i} \) is critically stable, and from the triangular structure of the Jacobian linearization of (44) it follows that the pair \((A_{x_i}, B_{x_i})\) admits a good saturated linear controller. Now, points 1, 2, and 3 of the induction hypothesis follow from the properties of \( k_i \), the properties of \( \sigma \), the fact that \( \Gamma + \Omega \) is invertible, and the fact that \( I \) is a good saturated linear controller for the Jacobian linearization of (44). For point 4, we first make a copy of the \( x_i \) subsystem, the state of which we will denote \( \tilde{x}_i \), i.e.,

\[
\tilde{x}_i = f_i(\tilde{x}_i, k_i(\tilde{x}_i, v), d) \quad \tilde{x}_i(0) = x_i(0).
\]

(46)

Let \((A_{i+1}, B_{i+1})\) represent the Jacobian linearization of (44), and let \( g_{i+1}(x_i, v, d) \) be such that

\[
A_{i+1} \dot{x}_{i+1} + B_{i+1} v + g_{i+1}(x_i, v, d) = f_{i+1}(x_{i+1}, k_i(x_i, v), d).
\]

(47)

It follows from (39) that \( g \) is independent of \( x_a \). Notice that

\[
\lim_{(x_i, v) \to 0} \left[ \frac{g_{i+1}(x_i, v, 0)}{x_i, v} \right] = 0.
\]

(48)

We now use Theorem 3 to get that with respect to \((v, d), (x_{i+1}, \tilde{x}_i)\) satisfies an asymptotic bound from \( \mathbb{R}^{n+m} \times X \) with
linear gains and nonzero channel one restriction. It follows that with respect to $(v, d)$, $(\xi, x)$ satisfies an asymptotic bound from $\mathbb{R}^n \times X$ with linear gains and nonzero channel one restriction.

C. Nonlinear Feedforward Systems

As a general application of the results in the previous subsection, we consider the control of nonlinear systems in so-called “feedforward” form. Special cases of this form, including linear systems with saturation, will be discussed later.

Theorem 5 (Feedforward Systems): Consider a locally Lipschitz system of the form

$$ \begin{align*}
\dot{x}_1 &= A_1 x_1 + f_1(x_2, \ldots, x_p, u, d) \\
\dot{x}_2 &= A_2 x_2 + f_2(x_3, \ldots, x_p, u, d) \\
&\vdots \\
\dot{x}_{p-1} &= A_{p-1} x_{p-1} + f_{p-1}(x_p, u, d) \\
\dot{x}_p &= f_p(x_p, u, d)
\end{align*} $$

(49)

with $x_i \in \mathbb{R}^n$, $x_p \in \mathcal{X}_p \subset \mathbb{R}^p$, and $u \in \mathbb{R}^m$. Assume

1) for (49) with $d = 0$ and $u = 0$ the origin is an equilibrium point, and its the Jacobian linearization with $u$ as input and $d \equiv 0$ is stabilizable;
2) $A_i$, for $i = 1, \ldots, p - 1$, is critically stable;
3) there exist $X_p \subset \mathcal{X}_p$ and $k : X_p \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying the induction hypothesis for the $x_p$ subsystem.

Then there exists a function $\alpha(x)$ with the same smoothness properties as $k$ (in point 3 above) such that (49) with $u = \alpha(x)$ is zero-input LES and its state satisfies an asymptotic bound with respect to $d$ with linear gain. The restriction is nonzero if the channel 2 restriction given in the induction hypothesis is nonzero.

Sketch of the Proof: The proof is constructed from repeated application of Theorem 4. Since the linearization of the full system is stabilizable, it follows from the triangular structure of the linear approximation that the linearization at each step must be stabilizable.

In the case where $f_p$ is independent of $d$, the following result is useful for generating a controller that satisfies the induction hypothesis for the $x_p$ subsystem from a controller that gives the zero-input LES property.

Lemma 4.3: If the origin of the system $\dot{x} = f(x, \alpha(x))$ is LES with basin of attraction $X$, then there exist a strictly positive real number $\delta$ and a smooth function $\psi : \mathbb{R}_{\geq 0} \to [0, 1]$ such that $\psi(x) = 1$ for all $x \in [0, \delta]$ and the state of the system $\dot{x} = f(x, \alpha(x) + \psi(|x|)\sigma)$ satisfies an asymptotic bound from $X$ with linear gain and nonzero restriction.

Proof: According to Lemma 3.2, there exist an open neighborhood $X$ of the origin and strictly positive real numbers $\Delta$ and $N$ such that state of the system $\dot{x} = f(x, \alpha(x) + w)$ satisfies an $\mathcal{L}_\infty$ bound from $X$ with gain $N$-Id and restriction $\Delta$. Without loss of generality, assume $X \subset X$. Pick $\delta$ and $\delta'$ such that $0 < \delta < \delta'$ and $\{x \in \mathbb{R}^n : |x| \leq \delta' \} \subset X$. Then, choose $\psi$ to be a smooth function taking values in $[0, 1]$ and satisfying $\psi(|x|) = 1$ when $|x| \leq \delta$ and $\psi(|x|) = 0$ when $|x| \geq \delta'$. It then follows that for all initial conditions in $X$, the state of the system $\dot{x} = f(x, \alpha(x) + \psi(|x|)\sigma)$ remains in $X$ for all $t \geq 0$ and the set $X$ is recurrent. Assume $\|v\|_a \leq \Delta/2$, for example. Then there exists a positive real number $T$ sufficiently large such that $x(T) \in X$ and $\inf_{0 \leq t \leq T} |x(t)| \leq \Delta$. It follows that for all $x(0) \in X$ and all $v$ such that $\|v\|_a \leq \Delta/2$, we have $\|x\|_a \leq N\|v\|_a$.

D. Linear Systems with Saturation

In this section, we discuss several results for linear systems with saturation that follow from Theorem 5.

Theorem 6 (Small Input Stabilizable Linear Systems): Consider the system

$$ \begin{align*}
\dot{x} &= Ax + B\sigma(u) + d \\
y &= Cx
\end{align*} $$

(50)

where $\sigma$ is a saturation function. Suppose $(A, B)$ is stabilizable and $A$ has no eigenvalues with strictly positive real part. Then there exists a smooth function $\alpha(x)$ and a positive real number $l$ such that $|\alpha(x)| \leq l|x|$ and (50) with $u = \alpha(x)$ is zero-input LES, and its state satisfies an asymptotic bound from $\mathbb{R}^n$ with linear gain and nonzero restriction.

Sketch of the Proof: Rewrite (50) in any coordinates so that the matrix $A$ is transformed into block upper triangular form, e.g., Jordan canonical form. Apply Lemma 4.1 to the last block to get that Assumption 3) of Theorem 5 holds. So the existence of $\alpha$ follows from Theorem 5 and the bound on $\alpha$ follows from Lemma 4.1, Theorem 4, and Definition 4.1.

Theorem 7 (Output Feedback and Robustness): Consider the locally Lipschitz system

$$ \begin{align*}
\dot{x} &= Ax + g(u, d) \\
y &= Cx
\end{align*} $$

(51)

Suppose the pair $(C, A)$ is detectable, the pair $(\frac{\partial g}{\partial u}|_{u=0, d=0}) =: (A, B)$ is stabilizable, and $A$ has no eigenvalues with strictly positive real part. Let $L$ be such that the matrix $A + LC$ is Hurwitz and let $\alpha(x)$ satisfy the properties of Theorem 6 for $(A, B)$ and some saturation function $\sigma$. Then there exists a strictly positive real number $\lambda$ such that (51) together with the controller

$$ \begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\
u &= \lambda \sigma(\hat{x}/\lambda)
\end{align*} $$

(52)

is zero-input LES and its state satisfies an asymptotic bound from $\mathbb{R}^n \times \mathbb{R}^n$ with linear gain and nonzero restriction.

Sketch of the Proof: Define $x_1 = \hat{x}$, $x_2 = x - \hat{x}$, and rewrite the composite system as

$$ \begin{align*}
\dot{x}_1 &= A_1 x_1 + B\lambda \sigma(\alpha(x_1/\lambda)) + y_2, \\
y_1 &= \lambda \sigma(\alpha(x_1/\lambda)) \\
\dot{x}_2 &= (A + LC)x_2 + g(y_1, d) - B y_1, \\
y_2 &= -LCx_2.
\end{align*} $$

(53)

Notice that $\lim_{|y_1| \to 0} \frac{|g(y_1, 0) - By_1|}{|y_1|} = 0$. The remainder of the proof then follows the lines of the proof for Theorem 3. The details are omitted due to space constraints.
Theorem 8 (Rate Saturation): Consider the system
\[ \begin{align*}
\dot{x} &= Ax + B\sigma_1(u) + d_1 \\
\dot{u} &= \sigma_2(u) + d_2
\end{align*} \tag{54} \]
where \( x \in \mathbb{R}^n, u, v \in \mathbb{R}^m, \) and \( \sigma_1 \) and \( \sigma_2 \) are saturation functions. Suppose the pair \((A, B)\) is stabilizable and \( A \) has no eigenvalues with strictly positive real part. Then there exists a smooth function \( \alpha(x, u) \) and a positive real number \( \ell \) such that \( |\alpha(x, u)| \leq \ell |(x, u)| \) and (54), with \( v = \alpha(x, u) \), is zero-input LES, and its state satisfies an asymptotic bound from \( \mathbb{R}^n \times \mathbb{R}^m \) with linear gain and nonzero restriction with respect to \( d = (d_1, d_2) \).

**Sketch of the Proof:** As was the case for Theorem 6, in the proper coordinates all of the assumptions of Theorem 5 are satisfied.

We are not in a position to state precise stability results for systems with time-delayed inputs since we are working with differential equations of the form \( \dot{x} = f(x, u) \) throughout this paper. Nevertheless, it is still possible to discuss the input–output behavior of such systems. For example, consider system (54) with the controller \( v = \alpha(x, u) \) of the preceding theorem, but (54) is modified so that there is a small positive time delay in the input \( u \) to the \( \dot{x} \) equation, i.e.,
\[ \begin{align*}
\dot{x}(t) &= Ax(t) + B\sigma_1(u(t - \delta)) + d_1 \\
\dot{u}(t) &= \sigma_2(x(t), u(t)) + d_2
\end{align*} \tag{55} \]
We can interpret this system in the spirit of Fig. 1. In particular, define the state of \( \Sigma_2 \) to be \((x, u)\) and write
\[ \Sigma_2: \begin{align*}
\dot{x}_2 &= \begin{cases} 
\dot{x} = Ax + B\sigma_1(u) + y_1 \\
\dot{u}_2 = u_2(u) 
\end{cases} \\
y_1 &= B[\sigma(y_2(t - \delta)) - \sigma(y_2(t))] + d_1 \tag{56} 
\end{align*} \]
and let \( \Sigma_1 \) be the system
\[ \begin{align*}
y_1 &= B|\sigma(y_2(t - \delta)) - \sigma(y_2(t))| + d_1 \tag{57} \end{align*} \]
Now, if we assume existence and uniqueness of solutions on \([0, \infty), \) there exists \( m > 0 \) such that
\[ \|y_1\| \leq \delta m \|u\| + \|d_1\|. \tag{58} \]
Moreover, from the properties of \( \sigma_2 \) and \( \alpha \)
\[ \|y_1\| \leq \delta m \min \{b, K_2 \|x_2\| + \|d_2\| + \|d_1\|\}. \tag{59} \]
Finally, using that for any two positive real numbers \( c \) and \( d, \) \( c + d \leq 2 \max \{c, d\} \), using the asymptotic bound on the state of the \( x_2 \) subsystem, and applying the calculations of Theorem 2, we get that for \( \delta \) sufficiently small the full state satisfies an asymptotic bound from \( \mathbb{R}^n \times \mathbb{R}^m \) with linear gain and nonzero restriction.

All of the results so far have applied only to linear systems with no open-loop eigenvalues having strictly positive real part. This restriction allowed us to establish an asymptotic bound from the entire state space. The more general case is addressed below.

Theorem 9 (Systems with Exponentially Unstable Modes): Consider a linear system with input saturation, expressed in the form
\[ \begin{align*}
\dot{x} &= \begin{bmatrix} A_{eu} & 0 \\
0 & A_{eu} \end{bmatrix} x + \begin{bmatrix} B_{eu} \sigma(u) \\
B_{eu} \sigma(u) \end{bmatrix} \tag{60} 
\end{align*} \]
where \( \sigma \) is a saturation function.\(^4\) Suppose that the linear approximation is stabilizable and that \( A_{eu} \) has no eigenvalues with strictly positive real part. If there exists a locally Lipschitz control \( k_{eu}(x_{eu}) \), differentiable at the origin, such that
\[ \begin{align*}
x_{eu} &= A_{eu}x_{eu} + B_{eu}\sigma(k_{eu}(x_{eu})) \tag{61} \end{align*} \]
is LES with basin of attraction \( X \), then there exists a locally Lipschitz control \( k(x) \), differentiable at the origin, so that (60) with \( u = k(x) \) is LES with basin of attraction \( \mathbb{R}^n \times X \).

**Sketch of the Proof:** Express the \( x_1 + \) subsystem in coordinates so that \( A_{eu} \) is transformed into block upper triangular form. Then the full system is in the form (49) with the \( x_{eu} \) subsystem playing the role of the \( x_p \) subsystem in (49). Use Lemma 4.3 to get that Assumption 3) of Theorem 5 is satisfied.

E. A Class of Partially Linear Composite Systems

As a final application, we will contribute to the large body of work on stabilization of systems formed by the cascade of a linear system with a zero-input GAS nonlinear system, i.e., a smooth system of the form
\[ \begin{align*}
\dot{x} &= f(x, z) \\
\dot{z} &= Az + Bu \tag{62} 
\end{align*} \]
where \( x \in \mathbb{R}^n, z \in \mathbb{R}^s, \) and the origin of the \( x \) subsystem is zero-input GAS. There are several interesting examples that demonstrate necessary conditions that must be imposed on the functional dependence of \( z \) in the \( \dot{x} \) equation if no extra growth assumptions are made on \( f; \) see [26], [18], [2], and [1], for example. In the result most similar to the one we will state, given in [18], it is assumed that there exists a dynamic compensator such that the state of (62) together with the state of the dynamic compensator can be written as
\[ \begin{align*}
\dot{x} &= f(x, 0) + g(x, \xi_0, \xi_1)\xi_1 \\
\dot{\xi}_0 &= A_0\xi_0 + A_1\xi_1 \\
\xi(t) &= v \tag{63} 
\end{align*} \]
where \( \xi_0 \in \mathbb{R}^m, \xi_1 \in \mathbb{R}^m, A_0 \) is critically stable, and the pair \((A_0, A_1)\) is stabilizable. In our result, we will relax the requirement on \( A_0, \) only requiring that it have no eigenvalues with strictly positive real part at the price of not being able to allow \( \xi_0 \) dependence in the \( \dot{x} \) equation. Similar conditions were imposed in [32] and [12] to achieve semiglobal stabilization (namely, for each arbitrarily large compact set, there exists a controller achieving LAS with domain of attraction containing

\(^4\) The subscript "eu" stands for "critically unstable" while "eu" stands for "exponentially unstable."
the given compact set). In [18, Proposition 4], it was shown that for the system
\[
\dot{x} = f(x, Cz) \\
\dot{z} = Az + Bu
\]
if the system \((A, B, C)\) is stabilizable, right invertible, and has no invariant zeros with strictly positive real part, then there exists a dynamic compensator such that (64) together with the dynamic compensator can be written as
\[
\dot{x} = f(x, \xi_1) \\
\dot{\xi}_o = A_o\xi_o + A_1\xi_1 \\
\xi_1^{(r)} = v
\]
where \(\xi_o \in \mathbb{R}^{n_x}, \xi_1 \in \mathbb{R}^{n_m}, A_o\) has no eigenvalues with strictly positive real part, and the pair \((A_o, A_1)\) is stabilizable. For simplicity, we will take (65) as our starting point and refer the reader to [18] for definitions of invariant zeros, right invertibility, and the resulting connections to (64).

Theorem 10: For (65), if the \(x\) subsystem is zero-input GAS, \(A_o\) has no eigenvalues with strictly positive real part and the pair \((A_o, A_1)\) is stabilizable, then the origin is globally asymptotically stabilizable.

Proof: We begin with the observation, from [24, Lemma 3.2], that there exists a matrix \(G(x)\) of smooth functions, invertible for all \(x \in \mathbb{R}^p\), satisfying
\[
|G(x)v| \leq |v| \quad \forall x \in \mathbb{R}^p \\
G(x) = I \quad \forall x \in \{ x \in \mathbb{R}^p : |x| \leq 1 \}
\]
and such that the state of the system
\[
\dot{x} = f(x, G(x)v)
\]
satisfies an \(a-L_{\infty}\) stability bound from \(\mathbb{R}^p\) with some gain \(\gamma\) and restriction \(\Delta = \infty\). We now claim that there exists a smooth function \(\alpha(\xi_o)\) so that the system
\[
\dot{x} = f(x, G(x)(\alpha(\xi_o) + d)) \\
\dot{\xi}_o = A_o\xi_o + A_1G(x)(\alpha(\xi_o) + d)
\]
is zero-input LAS and satisfies the asymptotic bound from \(\mathbb{R}^n \times \mathbb{R}^{n_x}\). If this claim is true, then using the terminology of partial feedback linearization (see [8]), the “output function” \(y = G^{-1}(x)\xi_1 - \alpha(\xi_o)\) has a well-defined global vector relative degree \((r, \ldots, r)\), and the coordinates \((x, \xi_o, y, \tilde{y}, \ldots, \tilde{y}^{(r-1)}))\) provide a global diffeomorphism, and, through a feedback transformation, we get the system
\[
\dot{x} = f(x, G(x)(\alpha(\xi_o) + y)) \\
\dot{\xi}_o = A_o\xi_o + A_1G(x)(\alpha(\xi_o) + y) \\
\tilde{y}^{(r)} = w.
\]
Then, since the \((x, \xi_o)\) subsystem is zero-input LAS and its state satisfies an asymptotic bound from \(\mathbb{R}^n \times \mathbb{R}^{n_x}\) with respect to \(y\), any \(w\) that globally asymptotically stabilizes the \((y, \tilde{y}, \ldots, \tilde{y}^{(r)})\) subsystem globally asymptotically stabilizes the full system.

We now establish the above claim. Recall that \(x\) in (67) satisfies an \(a-L_{\infty}\) stability bound with gain \(\gamma\) and restriction \(\Delta = \infty\). Let \(\sigma\) be a smooth saturation function characterized by \(K\) and \(b\), where \(\gamma(2b) \leq 1\), and let \(\alpha(\xi_o)\) and \(l\) satisfy the properties of Theorem 6 for \((A_o, A_1)\) and \(\sigma\). Let the linear gain be \(N \cdot I_d\) and let the nonzero restriction be \(\Delta\). We will show that the claim is satisfied with \(\alpha(\xi_o) = \sigma(\alpha(\xi_o))\). We know from Lemma 3.4 that, in (67) if \(v\) is essentially bounded, then
\[
\|x\|_a \leq \gamma(\|v\|_a).
\]
It follows that for the system:
\[
\dot{x} = f(x, G(x)(\sigma(\alpha(\xi_o)) + d))
\]
if \(d\) is essentially bounded
\[
\|x\|_a \leq \gamma(\|\sigma(\alpha(\xi_o)) + d\|_a) \\
\leq \max \{ \gamma(2\|\sigma(\alpha(\xi_o))\|_a), \gamma(2\|d\|_a) \} \\
\leq \max \{ \gamma(\min(2b, 2Kl)\|\xi_o\|_a), \gamma(2\|d\|_a) \}.
\]
On the other hand, for the system
\[
\dot{\xi}_o = A_o\xi_o + A_1\sigma(\alpha(\xi_o)) + A_1[(G(x) - I)\sigma(\alpha(\xi_o)) + G(x)d]
\]
if \(\|x\|_a \leq 1\) and \(\|d\|_a \leq \Delta\), then \(\|\xi_o\|_a \leq N\|d\|_a\). So, the asymptotic bound in the claim follows from Theorem 2. The zero-input LAS property in the claim follows from Theorem 1 since there exist strictly positive real numbers \(N_o, \Delta_o\) such that the state \(\xi_o\) of (73) satisfies an \(a-L_{\infty}\) stability bound from an open neighborhood of the origin with gain \((0, I_d, N_o \cdot I_d)\) and restriction \((1, \Delta_o)\) with respect to \((x, d)\).

F. Examples of Feedforward Systems

Example 4.1—Inverted Pendulum on a Cart Consider the inverted pendulum on a cart (assuming point masses, a massless rod, etc.)
\[
\dot{x} = \frac{1}{(M/m) + \sin^2(\theta)} \left[ \frac{u}{m} + \theta^2 \sin(\theta) - g \sin(\theta) \cos(\theta) \right] \\
\dot{\theta} = \frac{1}{l(M/m) + \sin^2(\theta)} \left[ \frac{u}{m} \cos(\theta) \right. \\
\left. - \theta^2 \cos(\theta) \sin(\theta) + \frac{m + M}{m} g \sin(\theta) \right]
\]
where \(x\) represents the position of the cart, \(\theta\) represents the angle of the pendulum with \(\theta = 0\) representing the inverted position. \(M\) is the mass of the cart, and \(m\) is the (point) mass at the end of the rod which has length \(l\). The variable \(u\) is the control input force acting on the cart. We invert the relationship between the input \(u\) and the acceleration of the cart, which can be done globally in the state space, yielding the control system
\[
\begin{align}
\dot{x} &= v \\
\dot{\theta} &= \frac{g}{l} \sin(\theta) - \frac{1}{l} \cos(\theta) v.
\end{align}
\]
If we can find a smooth function \( k_2(\theta, \dot{\theta}) \) which makes the \((\theta, \dot{\theta})\) subsystem LES with basin of attraction \( \mathcal{X}_2 \), then using Lemma 4.3, all of the assumptions of Theorem 5 will be satisfied and we will be able to achieve LES for the full system with basin of attraction \( \mathbb{R}^2 \times \mathcal{X}_2 \). Define
\[
\mathcal{X}_2 := \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \mathbb{R}
\]
and choose \( k_2 \) as
\[
k_2(\theta, \dot{\theta}) = \frac{1}{\cos(\theta)} \left[ \frac{g}{l} \sin(\theta) + \tan(\theta) + c \sigma(\dot{\theta}) \right]
\]
with \( c > 0 \), \( \sigma(s) > 0 \) for \( s \neq 0 \), \( \frac{\partial \sigma(s)}{\partial s} \big|_{s=0} > 0 \). Observe that this control is smooth on the set \((\theta, \dot{\theta}) \in \mathcal{X}_2\). Then, on this set, the closed-loop dynamics are governed by
\[
\ddot{\theta} = -\tan(\theta) - c \sigma(\dot{\theta}).
\]
Now consider the Lyapunov function candidate
\[
V(\theta, \dot{\theta}) = \int_0^\theta \tan(s) \, ds + \frac{1}{2} \dot{\theta}^2 = \ln|\cos^{-1}(\theta)| + \frac{1}{2} \dot{\theta}^2.
\]
Notice that \( V \to \infty \) as \( |\theta| \to \frac{\pi}{2} \) or as \( |\theta| \to \infty \). Taking the derivative of this function along the vector field in (78) gives
\[
\dot{V} = \tan(\theta) \dot{\theta} + [\tan(\theta) - c \dot{\theta} \sigma(\dot{\theta})] = -c \dot{\theta} \sigma(\dot{\theta}).
\]
Since \( c > 0 \) and \( \sigma(s) > 0 \) for all \( s \neq 0 \), it follows that the origin is stable and all trajectories which start in \( \mathcal{X}_2 \) remain in that set thereafter. LaSalle’s invariance principle gives that the origin is LAS with basin of attraction \( \mathcal{X}_2 \). The LES property follows from the linearization since \( \sigma'(0) > 0 \). It is also interesting to consider the linear approximation of the inverted pendulum on a cart with input saturation. The linearization has one open-loop eigenvalue with strictly positive real part. Using the algorithm suggested in this paper, we can achieve the maximal possible domain of attraction with a smooth control. Because of space constraints, we do not discuss this problem in more detail. This example is discussed further in [33].

**Example 4.2**—The PVTOL Aircraft [6]: Consider the equations of motion for the PVTOL
\[
\begin{align*}
\dot{x} &= -\sin(\theta)u_1 + \epsilon \cos(\theta)u_2 \\
\dot{y} &= \cos(\theta)u_1 + \epsilon \sin(\theta)u_2 - 1 \\
\dot{\theta} &= u_2
\end{align*}
\]
where \( x \) is the horizontal position of the aircraft, \( y \) is the vertical position of the aircraft, and \( \theta \) is the angle the aircraft makes with the horizon. The variable \( u_1 \) is the thrust input and \( u_2 \) is the angular acceleration input. The parameter \( \epsilon \) is small, typically unknown, and characterizes the coupling between the rolling moment and lateral acceleration. We are interested in regulating the aircraft to some fixed position \((x_d, y_d)\) with \( \theta_d = 0 \). In [16] it was shown that this system is dynamically feedback linearizable regardless of the value of \( \epsilon \). Accordingly, it is possible to use dynamic feedback linearization to stabilize to an arbitrary point from a large region of the state space. Our goal here is to present an alternative control algorithm for this goal which, as we will establish through simulation, is apparently more robust to input corruption.
The system of (81) is in feedforward form (after introducing the appropriate constant offset in $u_1$) regardless of the value of $\epsilon$. The $(y, \dot{y}, \theta, \dot{\theta})$ subsystem plays the role of the $x_p$ subsystem in (49). Moreover, using the control

$$
\begin{align*}
    u_1 &= \frac{1}{\cos(\sigma_a(\theta))} [1 - \epsilon \sin(\theta)] u_2 - (y - y_d) - 2\dot{y} + v_1 \\
    u_2 &= -\theta - 2\dot{\theta} + v_2 \\
\end{align*}
$$

(82)

where $\sigma_a(s) = \text{sgn}(s) \min \{|s|, a\}$ and $a < \frac{\pi}{2}$, the $(y, \dot{y}, \theta, \dot{\theta})$ subsystem is zero-input LES and satisfies an asymptotic bound with respect to $v = (v_1, v_2)$ from $\mathbb{R}^2$ with linear gain and nonzero restriction. (This can be established using Theorem 2.) Thus, we conclude from Theorem 5 that the origin is globally asymptotically stabilizable. Defining $\xi = [(x - x_d) \ x \ \theta \ \dot{\theta}]^T$, the control law can be chosen to be of the form

$$
\begin{align*}
    u_1 &= \frac{1}{\cos(\sigma_a(\theta))} [1 - \epsilon \sin(\theta)] u_2 - (y - y_d) - 2\dot{y} \\
    u_2 &= -\theta - 2\dot{\theta} + \sigma_1(F_1\xi + \sigma_2(F_2\xi)) \\
\end{align*}
$$

(83)

since the linear approximation of the system (81) and (82) is stabilizable even when $v_1 \equiv 0$. The matrix $F_1$ should be a good saturated linear controller for the $(x, \theta, \dot{\theta})$ subsystem of (81) and (82) with $y = \dot{y} = v_1 \equiv 0$ and similarly for $F_2$ with respect to $(x - x_d, \dot{x}, \theta, \dot{\theta})$.  

While, in principal, we need knowledge of $\epsilon$ in our controller, according to Remark 4.2 we might expect some robustness to uncertainty in $\epsilon$. Similarly, although we have not designed our controller to explicitly account for magnitude or rate saturations (we could have since this would still be a system in feedforward form) and time delays, since the $x - x_d$ and $\dot{x}$ dependence in our controller is bounded, we might expect the stability achieved by our controller to be robust to this type of input corruption at least as long as $(\theta, \dot{\theta}, y - y_d, \dot{y})$ start off sufficiently small. To corroborate these expectations, we designed a controller assuming $\epsilon = 0$, and we ran simulations where $\epsilon = 0.1$ and where the actual inputs, called $u_{1\epsilon}$ and $u_{2\epsilon}$, respectively, satisfied $u_{1\epsilon}(t) = \text{sat}_2(u_1(t - 0.3))$, i.e., $u_1$ is subject to magnitude saturation and time delay, and $u_{2\epsilon} = \text{sat}_5(\dot{z})$ where $\dot{z} = 10\text{sgn}(-z + u_2)$, i.e., $u_2$ is subject to magnitude and rate saturation. The controller was given by (83) with $\epsilon = 0$, $a = 1.2$, $F_1 = (0, -1, 2, 1)$, $F_2 = (-1, -4, 5, 2)$, $\sigma_1(s) = \text{sgn}(s) \min \{|s|, 1\}$, and $\sigma_2(s) = \text{sgn}(s) \min \{|s|, 10\}$. We started the PVTOL at the position $(x, y) = (200, 5)$ and asked the controller to move the PVTOL to the position $(x, y) = (0, 5)$. The simulation results are shown in Fig. 2. We compared this to a dynamic feedback linearizing controller where again $\epsilon = 0$ was assumed for control design, where the linear approximation of the closed loop was essentially the same as that using the saturating controller, and running simulations with $\epsilon = 0.1$ and the same input corruption. The results are shown in Fig. 3. \hfill $\Box$

V. CONCLUSION

In this paper we have presented a novel small gain theorem, and we have used it extensively for control design. In particular, we have established a stabilization algorithm for nonlinear systems in so-called feedforward form. Special cases
of systems in this form include stabilizable linear systems with input magnitude and rate saturation. We have also discussed the stabilization problem for two specific linear systems in feedforward form: the inverted pendulum on a cart and the VTOL aircraft.

The work in this paper is meant to illustrate how input-output stability results serve to complement Lyapunov function stability results in the box of tools useful for nonlinear control design. Other places where this point has recently been made include [9] and [3].

Regarding control of the nonlinear systems, it seems that identifying the structure of the particular system to be controlled is crucial. It will be interesting to see how many practical systems can be controlled successfully using the ideas applicable to nonlinear feedforward systems.

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APPENDIX

THEOREM 1, FINITE $\Delta_1^{(1)}$ OR $\Delta_1^{(2)}$

This case is complicated by the fact that when $d$ is only measurable and $\Delta_1^{(1)} < \infty$, for example, the existence of a finite $T > 0$ such that $\sup_{t \in [0,T]} |y_2(t)| < \Delta_1^{(1)}$ does not necessarily imply that there exists $T' > T$ such that $\sup_{t \in [0,T']} |y_2(t)| < \Delta_1^{(1)}$. (When $\Delta_1^{(1)} = \infty$, it does follow by using Lemma 3.1 and continuity of $x$ and $h$.) So we will proceed differently, exploiting the assumption that $X_1 \times \hat{X}_2$ is path connected and that the output is zero for all positive time when the initial condition is the origin and $d \equiv 0$.

Given $x_o$, let $p(x_o, \lambda)$ represent a continuous path in $X_1 \times \hat{X}_2$ from the origin to $x_o$ with the property that $p(x_o, 0)$ is the origin and $p(x_o, 1) = x_o$, and let $y_1^\lambda$ and $y_2^\lambda$ represent the outputs produced starting at $x_0^\lambda := p(x_o, \lambda)$ with input $\lambda d$. Note that when $\lambda = 0$, the solutions are defined on $[0, \infty)$ and the outputs are identical zero. Note also that the solutions are a continuous function of $\lambda$. In other words, given $T > 0$ (arbitrarily large) and given $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists $\lambda^* > 0$ such that $\lambda \in [0, \lambda^*]$ implies that the solution exists on $[0, T]$ and

$$\|y_1^\lambda\|_{\infty} \leq \epsilon_2, \quad \|y_2^\lambda\|_{\infty} \leq \epsilon_1$$

(see [10, Th. 2.6], for example). Let

$$\Delta_1^{(2)}(\lambda) = \max \left\{ \gamma_0^{(1)} \left( \max_{\lambda \in [0,1]} |x_1^\lambda| \right), \gamma_1^{(1)} \circ \gamma_0^{(2)} \left( \max_{\lambda \in [0,1]} |x_2^\lambda| \right), \gamma_1^{(1)} \circ \gamma_2^{(2)} (\|d_1^\lambda\|_{\infty}), \gamma_2^{(2)} (\|d_2^\lambda\|_{\infty}) \right\}$$

and

$$\Delta_1^{(3)}(\lambda) = \max \left\{ \gamma_0^{(1)} \left( \max_{\lambda \in [0,1]} |x_1^\lambda| \right), \gamma_1^{(1)} \circ \gamma_0^{(2)} \left( \max_{\lambda \in [0,1]} |x_2^\lambda| \right), \gamma_1^{(1)} \circ \gamma_2^{(2)} (\|d_1^\lambda\|_{\infty}), \gamma_2^{(2)} (\|d_2^\lambda\|_{\infty}) \right\}$$

and note that $\Delta_1^{(1)} < \Delta_1^{(2)}$ and $\Delta_1^{(2)} < \Delta_1^{(3)}$ [see (11) and (12)]. Now, let $\epsilon_1$ and $\epsilon_2$ satisfy $\Delta_1^{(1)} < \epsilon_1 < \Delta_1^{(3)}$, $\epsilon_2 < \epsilon_2 < \Delta_1^{(2)}$, and let $\lambda^*$ be the largest value belonging to the interval $[0, 1]$ such that (84) holds for all $\lambda \in [0, \lambda^*]$. Suppose $\lambda^* < 1$. Then we have, using the same calculations as for the case where $\Delta_1^{(1)}$ and $\Delta_1^{(2)}$ were infinite, that

$$\|y_1^{\lambda^*}\|_{\infty} \leq \Delta_1^{(1)} < \epsilon_2, \quad \|y_2^{\lambda^*}\|_{\infty} \leq \Delta_1^{(2)} < \epsilon_1.$$ 

By continuity of solutions, there exists $\lambda > \lambda^*$ so that (84) holds, thus contradicting that $\lambda^* < 1$. We conclude that $\lambda^* = 1$ and, since $T$ is arbitrary, the solutions are defined on $[0, \infty)$ and $\|y_1\|_{\infty} < \Delta_1^{(1)}$, $\|y_2\|_{\infty} < \Delta_1^{(2)}$. The remainder of the proof is the same as for the case where $\Delta_1^{(1)}$ and $\Delta_1^{(2)}$ are infinite.

REFERENCES


Andrew R. Teel (S’91–M’92), for a photograph and biography, see p. 378 of the March 1996 issue of this TRANSACTIONS.