Filter Banks - V
Paraunitary Perfect Reconstruction Filter Banks

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Paraunitary Perfect Reconstruction Filter Banks.

Introduction:

Analysis and synthesis filter banks of $M$ channel maximally decimated filter bank can be expressed in terms of polyphase matrices $E(z)$ and $R(z)$. Such a filter bank with FIR filters has ‘perfect reconstruction’ property iff $E(z)$ is just a delay. i.e.

$$\det E(z) = \alpha z^{-K}.$$  

We shall discuss Perfect Reconstruction filter banks in which the polyphase matrix $E(z)$ satisfies a special property called the lossless or Paraunitary property

- Synthesis filter and analysis filters have the same length

- This property is basic to the generation of the “Orthonormal Wavelet basis “
Paraunitary Property

In our earlier discussions, the analysis bank is described by an $M \times 1$ transfer matrix $h(z)$ and the synthesis filter by $1 \times M$ transfer matrix $f^T(z)$ which are expressed in terms of polyphase matrices $E(z)$ and $R(z)$ as:

$$h(z) = E(z^M) \cdot e(z) \quad f^T(z) = z^{-(M-1)} e^-(z) R(z^M) \quad .... \ (1)$$

Lossless transfer Matrix:

A $p \times r$ causal matrix $H(z)$ is said to be lossless if

- each entry $H_{km}(z)$ is stable
- $H(e^{jw})$ is unitary, that is,

$$H^\bullet(e^{jw}) \cdot H(e^{jw}) = dI_r \quad (H^\bullet(e^{jw}) \text{ is transpose-conjugate of } H(e^{jw}) \quad ....\ (2)$$

“$H(z)$ is lossless” is equivalent to “the LTI system with transfer function $H(z)$ is lossless.”
Paraunitary Property

Equation (2) is called Unitary property. Thus $H(z)$ is unitary on the unit circle in the $Z$ plane. In order to satisfy (2), $p > r$. The subscript ‘$r$’ in $I_r$ means that it is a $r \times r$ matrix.

Paraunitary Property:

For rational transfer functions, (2) implies that:

$$\tilde{H}(z)H(z) = dI,$$ for all $z$  

….. (3)

This is termed as paraunitary property.

(Note: $\tilde{H}(z)$ is the complex conjugate of $H(z)$

i.e. $\tilde{H}(z) = H^*(z)$

Ex. :- Let $H(z) = (a + bz^{-1})$, then

$$\tilde{H}(z) = a^* + b^* z$$

So, for a causal system to be lossless, it is sufficient to prove

* stability
* paraunitariness.
Paraunitary property

Observations:

1. If $\mathbf{H}(z)$ is square and lossless, then $\tilde{\mathbf{H}}(z)$ is paraunitary but not lossless (unless it is a constant).

2. “Lossless” and “Paraunitariness” are used interchangeably.

Normalized systems: If a lossless system has $d = 1$, then we say it as normalized-lossless.

Square matrices:

For square matrices, equation (2) implies that

$$\mathbf{H}^{-1}(z) = \tilde{\mathbf{H}}(z) / d$$

i.e. the inverse of the matrix can be obtained by use of ‘tilde’ operation.

In this case, every row is power complementary, and any pair of rows is orthogonal since

$$\tilde{\mathbf{H}}(z) \cdot \mathbf{H}(z) = \mathbf{H}(z) \cdot \tilde{\mathbf{H}}(z) = d \mathbf{I}$$
Properties of Paraunitary Systems.

(Note: Power complimentary transfer functions:

\[ H_0(z), H_1(z) \] are said to be power complimentary if

\[
\left| H_0(e^{j\theta}) \right|^2 + \left| H_1(e^{j\theta}) \right|^2 = c^2 \quad \text{for all } \theta
\]

Some properties of Paraunitary Systems:

1. **Determinant is allpass.** For a square matrix, \(|H(z)|\) is all pass, in particular, if \(H(z)\) is FIR then, \(|H(z)|\) is a delay i.e.

\[
\det H(z) = a z^{-K}, \quad K \geq 0, \quad a \neq 0
\]

2. **Power Complimentary Property.** For a M x 1 transfer matrix \(h(z) = [H_0(z) \ldots H_{M-1}(z)]\), then

\[
\sum_{k=0}^{M-1} \left| H_k(e^{j\theta}) \right|^2 = c \quad \text{for all } \theta
\]

3. **Submatrices of paraunitary** \(H(z)\). Every column of a paraunitary transfer matrix is itself paraunitary.
Examples:

1. \( \mathbf{K}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \) then we have

\[
\tilde{\Lambda}(z) \Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} = I
\]

Therefore, \( \mathbf{K}(z) \) is paraunitary.

2. The system in the adjacent figure has a transfer matrix

\[
e(z) = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}
\]

\[
\tilde{e}(z) = \begin{bmatrix} 1 & z \end{bmatrix}
\]

Therefore, \( \tilde{e}(z) \cdot e(z) = \begin{bmatrix} 1 & z \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} = 2 \)

So, \( e(z) \) is paraunitary!
3. Paraunitary Filter banks:

Consider the system in the figure, which is a cascade of two systems with transfer matrices \( e(z) \) and \( W^\ast \) respectively.

\( W \) is an \( M \times M \) DFT matrix which is unitary, we have already seen that \( e(z) \) is also paraunitary in previous example, i.e.

\[
\tilde{e}(z).e(z) = M
\]

Thus, overall transfer matrix is also paraunitary.

\[
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
\vdots \\
H_{M-1}(z)
\end{bmatrix} = W^\ast e(z)
\]
Filter Bank Properties

From the previous discussion, the paraunitary property implies

\[ \tilde{E}(z)E(z) = d I, \]  
that is, \( E^t(z) = E(z) / d \) for all \( z \)

So we choose \( R(z) \) as

\[ R(z) = cz^{-K} \tilde{E}(z) \]  
….. (5)

Note: Positive \( K \) ensures that \( R(z) \) is causal.

Stability:

If the analysis filters are stable and IIR, then choice of \( R(z) \) as per equation (3) results in unstable filters!

So, we cannot build useful Perfect Reconstruction Systems with IIR lossless \( E(z) \)!!

Hence we restrict our attention to \( \text{FIR} \) filters.
Properties

1. Relation between Analysis and Synthesis Filters:

Substituting equation (5) in equation (1) for synthesis filters, we obtain

\[
\begin{align*}
{f^T}(z) &= z^{-(M-1)} e^\sim(z) R(z^M) \\
      &= c z^{-(M-1+MK)} e^\sim(z) E^\sim(z^M) \\
      &= c z^{-(M-1+MK)} h^\sim(z). \\
\end{align*}
\]

Let \( L = M-1+MK \)

\[
F_k(z) = c z^{-L} \tilde{H}_k(z)
\]

….. (6)

In time domain, it can be expressed as:

\[
f_k(n) = c h_k^\star(L - n), \quad 0 < k < M-1
\]

In frequency domain, it implies that

\[
|F_k(z)| = |c| |H_k(e^{i\omega})|
\]

i.e. the magnitude responses of \( F_k(z) \) and \( H_k(z) \) are exactly the same (with a scale factor \( c \))
Properties

Theorem:

Consider a maximally decimated QMF bank with causal FIR analysis filters $H_k(z)$, and let $E(z)$ be the polyphase matrix for the analysis filters. Then,

1. $E(z)$ is lossless (that is, paraunitary)
2. The synthesis filters are given by $f_k(n) = c h^*_k (L - n), \quad 0 \leq k \leq M-1$
3. The system has perfect reconstruction property.

2. Power Complimentary property:

Consider the vector of analysis filters $h(z) = E(z^M) e(z)$. $h(z)$ is paraunitary which implies that analysis filters $H_k(z)$ are power complimentary i.e.

$$\sum_{k=0}^{M-1} \left| H_k(e^{j\theta}) \right|^2 = \text{positive constant}$$
3. **AC matrix is paraunitary if and only if $E(z)$ is Paraunitary.**

4. **Relation to Mth band filters**

   If $E(z)$ is Paraunitary, the each analysis filter $H_k(z)$, is a spectral factor of a (Zero phase) Mth band filter. The filter $G_k(z)$, defined as:

   $$ G_k(z) \cong \tilde{H}_k(z) \cdot H_k(z) $$

   is an Mth band filter.
Two Channel FIR paraunitary QMF Banks

Consider a two channel QMF filter bank with causal FIR filters given by

\[ H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n} \quad \quad H_1(z) = \sum_{n=0}^{N} h_1(n)z^{-n} \]

The alias-component matrix (AC) is given by:

\[ H(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \]

Paraunitariness of \( H(z) \) implies that \( \tilde{H}_k(z).H_k(z) = I, \) where \( \mathcal{J} = 2d. \)

From this, we obtain:

\[ \tilde{H}_0(z).H_0(z) + \tilde{H}_0(-z).H_0(-z) = \mathcal{J} \quad \quad \ldots \quad (7 \; a) \]
\[ \tilde{H}_1(z).H_1(z) + \tilde{H}_1(-z).H_1(-z) = \mathcal{J} \quad \quad \ldots \quad (7 \; b) \]
\[ \tilde{H}_0(z).H_1(z) + \tilde{H}_0(-z).H_1(-z) = 0 \quad \quad \ldots \quad (7 \; c) \]
Two Channel FIR paraunitary QMF Banks

The above equations imply that

\[ \left| \tilde{H}_0(z) . H_0(z) \right|_{\downarrow 2} = 0.5 \] \[ \left| \tilde{H}_1(z) . H_1(z) \right|_{\downarrow 2} = 0.5 \] \[ \left| \tilde{H}_0(z) . H_1(z) \right|_{\downarrow 2} = 0 \quad \ldots (8) \]

From this, we can say that:

- \( \tilde{H}_0(z) . H_0(z) \) is a half-band filter, i.e. \( H_0(z) \) is power symmetric.
- Order of \( H_0(z) \) is necessarily odd, \( N = 2J + 1 \)

Relation between the Two Analysis Filters:

From equation (7 b) we have that

\[ \frac{H_1(z)}{H_0(z)} = -\tilde{H}_0(-z) \]

\[ \frac{\tilde{H}_1(z)}{\tilde{H}_0(-z)} \]
Two Channel FIR paraunitary QMF Banks

From equation (7a) we have that

\[ \tilde{H}_0(z).H_0(z) + \tilde{H}_0(-z).H_0(-z) = f \]

which implies that there are no common factors between \( H_0(z) \) and \( H_1(z) \) (since right hand side is a constant).

Hence we conclude that

\[ H_1(z) = cz^{-1}\tilde{H}_0(-z) \]

\[ \cdots (9) \]

This is equivalent to in frequency domain as:

\[ |H_1(e^{j\xi})| = |H_0(-e^{j\xi})| = |H_0(e^{j(\xi-\nu)})| \]

i.e. the magnitude response of \( H_1(z) \) is obtained by shifting that of \( H_0(z) \) by \( \pi \).

For a real coefficient case, this means that if \( H_0(z) \) is low-pass then \( H_1(z) \) is high-pass both filters have the same ripple sizes, and same transition band-widths.
Two Channel FIR paraunitary QMF Banks

Design of Perfect Reconstruction QMF bank:

• First design a zero-phase half-band filter $H(z)$ with $H(e^{jw}) \geq 0$.
• Compute the spectral factor $H_0(z)$ (see section 3.2.5 or appendix D of text) which gives one of the analysis filters with order $N = 2J + 1$.
• Obtain the other analysis filter $H_1(z)$ and the two synthesis filters $F_0(z), F_1(z)$ as:

$$H_1(z) = -z^{-N} \tilde{H}_0(-z), \quad F_0(z) = z^{-N} \tilde{H}_0(z), \quad \text{and} \quad F_1(z) = z^{-N} \tilde{H}_1(z) \quad \text{.. (10)}$$

Equivalently, the above expression can be written as:

$$h_1(n) = (-1)^n h_0^*(N-n) \quad f_0(n) = h_0^*(N-n), \quad \text{and} \quad f_1(n) = h_1^*(N-n) \quad \text{.. (11)}$$
Two Channel FIR paraunitary QMF Lattice

FIR Two channel QMF bank with real coefficients:

Consider the following cascaded structure.

\[
R_m(z) = \begin{bmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{bmatrix} \quad \theta_m \text{ is real}
\]

* \(R_m\) is unitary

\(c\) denotes \(\cos(\theta_m)\) and \(s\) denotes \(\sin(\theta_m)\)
Two Channel FIR paraunitary QMF Lattice

Any 2x2 real coefficient (causal, FIR) paraunitary matrix can be factored as:

\[ E(z) = aR_j \Lambda(z)R_{j-1} \ldots \Lambda(z)R_0 \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \]

where \( \Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \), \( a \) is a positive scalar.

Synthesis bank which would result in perfect reconstruction is given by applying equation (5):

\[ R(z) = a \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix} R_0^T \Gamma(z) \ldots R_{j-1}^T \Gamma(z)R_j \]

where, \( \Gamma(z) = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \).
Two Channel FIR paraunitary QMF Lattice

Lattice structure for synthesis filter bank:

Analysis and Synthesis filters have order $N = 2J + 1$. 
Two Channel FIR paraunitary QMF Lattice

In a more efficient lattice structure, the rotation matrix $R_m$ can be written as:

$$R_m = \cos \theta_m \begin{bmatrix} 1 & a_m \\ -a_m & 1 \end{bmatrix} \quad \text{if } \cos \theta_m \neq 0$$

$$R_m = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{otherwise.}$$

Lattice structure can now be redrawn as shown in the following figure with $S = \alpha \prod_m \cos \theta_m$. 
Two Channel FIR paraunitary QMF Lattice

Analysis Bank

Synthesis Bank
(1 + a_m^2) H_0^{(m-1)}(z) = H_0^{(m)}(z) - a_m H_1^{(m)}(z),
(1 + a_m^2) z^{-2} H_1^{(m-1)}(z) = a_m H_0^{(m)}(z) + H_1^{(m)}(z)

\textit{Schematic for the } m\text{-th stage}

The } m\text{th stage filters can be obtained from } m-1 \text{ th stage filters as:}

\begin{align*}
H_0^{(m)}(z) &= H_0^{(m-1)}(z) + a_m z^{-2} H_1^{(m-1)}(z), \\
H_1^{(m)}(z) &= -a_m H_0^{(m-1)}(z) + z^{-2} H_1^{(m-1)}(z)
\end{align*}

\textit{.. (12) \textit{.. (13)}}

The coefficients } \alpha_m \text{ are calculated by inverting the above equations to obtain:
Derivation of $13$ from $12$:

From $12$

\[
H_0^{(m)}(z) = H_0^{(m-1)}(z) + a_m z^{-2} H_1^{(m-1)}(z),
\]

\[
H_1^{(m)}(z) = -a_m H_0^{(m-1)}(z) + z^{-2} H_1^{(m-1)}(z)
\]

which can be written as:

\[
\begin{bmatrix}
H_0^{(m)}(z) \\
H_1^{(m)}(z)
\end{bmatrix} =
\begin{bmatrix}
1 & a_m z^{-2} \\
-a_m & z^{-2}
\end{bmatrix}
\begin{bmatrix}
H_0^{(m-1)}(z) \\
H_1^{(m-1)}(z)
\end{bmatrix}
\]

\[
\Rightarrow
\begin{bmatrix}
H_0^{(m-1)}(z) \\
H_1^{(m-1)}(z)
\end{bmatrix} =
\begin{bmatrix}
1 & a_m z^{-2} \\
-a_m & z^{-2}
\end{bmatrix}^{-1}
\begin{bmatrix}
H_0^{(m)}(z) \\
H_1^{(m)}(z)
\end{bmatrix}
\]

\[
\Rightarrow
\begin{bmatrix}
H_0^{(m)}(z) \\
H_1^{(m)}(z)
\end{bmatrix} =
\frac{1}{a_m^2 z^{-2} + z^{-2}}
\begin{bmatrix}
z^{-2} - a_m z^{-2} \\
a_m & 1
\end{bmatrix}
\begin{bmatrix}
H_0^{(m-1)}(z) \\
H_1^{(m-1)}(z)
\end{bmatrix}
\]

\[
\Rightarrow
(1 + a_m^2) z^{-2}
\begin{bmatrix}
H_0^{(m)}(z) \\
H_1^{(m)}(z)
\end{bmatrix} =
\begin{bmatrix}
z^{-2} - a_m z^{-2} \\
a_m & 1
\end{bmatrix}
\begin{bmatrix}
H_0^{(m-1)}(z) \\
H_1^{(m-1)}(z)
\end{bmatrix}
\]

From which it follows that:

\[
(1 + a_m^2) H_0^{(m-1)}(z) = H_0^{(m)}(z) - a_m H_1^{(m)}(z),
\]

\[
(1 + a_m^2) z^{-2} H_1^{(m-1)}(z) = a_m H_0^{(m)}(z) + H_1^{(m)}(z)
\]
Two Channel FIR paraunitary QMF Lattice

Properties of Paraunitary QMF Lattice:

The properties of the QMF lattice are almost similar to that discussed in previous section(s).

Completeness:

- Every two channel (real coefficient, FIR) paraunitary QMF bank can be represented using the above lattice structure.

- We can always define \( H_1(z) = -z^{-N} H_0(-z^{-1}) \) and implement the analysis bank using the above lattice, given a real coefficient power symmetric FIR filter \( H_0(z) \)
Complexity of Paraunitary QMF lattice:

The total number of multipliers required to implement the lattice sections in the analysis is equal to $2(J + 1) + 2$.

Each of these operates at half the input sampling rate, so that we have an average of $J + 2$ MPU’s.

Therefore, MPU’s to implement the lattice sections in analysis bank $= J + 2 = 0.5(N + 3)$.

Each lattice section requires two additions, so $J + 1$ sections require $2(J + 1)$ additions and each operate at half the input sampling rate.

Therefore, total number of APU’s $= (J + 1) = 0.5(N + 1)$.

Synthesis bank has the same complexity.

Thus, lattice structure is more efficient, requiring only half as many MPU’s as the direct form!