SUMMARY OF IIR FILTER THEORY

Yogananda Isukapalli
Analog Frequency Specifications

\[ |H(j \omega')| \]

Amplitude

\[ 1 + \delta_p \leq |H(j \omega')| \leq 1 + \delta_p \text{ for } |\omega'| \leq \omega'_p, \]

\[ |H(j \omega')| \leq \delta_s \text{ for } \omega'_s \leq |\omega'| \leq \infty, \]

\[ \alpha_p = -20 \log_{10} (1 - \delta_p) dB, \]

\[ \alpha_s = -20 \log_{10} (\delta_s) dB, \]

where \( \alpha_p \) and \( \alpha_s \) are the peak pass band ripple and minimum stop band attenuation in dB respectively.
Analog Filter

\[ y(t) = \sum_{k=0}^{N} a_k \frac{d^k x(t)}{dt^k} + \sum_{k=0}^{M} b_k \frac{d^k y(t)}{dt^k} \]

**Transfer function**

\[
H(s) = \frac{\sum_{i=0}^{N} a_i s^i}{1 - \sum_{i=0}^{M} b_i s^i}
\]

**Impulse response**

\[
h(t) = L^{-1} H(s)
\]

**Example:**

\[
H(s) = \frac{a_0}{1 - b_1 s}
\]

\[
h(t) = \begin{cases} 
\frac{a_0}{b_1} e^{b_1 t} & \text{for } t \geq 0 \\
0 & \text{for } t < 0
\end{cases}
\]
Analog filters are stable when their poles are in the left half of the s-plane, whereas digital filters are stable when their poles are confined within the unit circle.

**Stability**

**Analog Filter : Magnitude**

Function: \[ H(j\omega') = H(s) |_{s=j\omega'} = |H(j\omega')| < H(j\omega') \]

Magnitude Square Function: \[ |H(j\omega')|^2 = H(s)H(-s) |_{s=j\omega'} \]
Analog Butterworth Filters

Define: Lowpass Butterworth Filters (N^{th} order)

\[ |H(j\omega')|^2 = \frac{1}{1 + (\frac{\omega'}{\omega'_p})^{2N}} \]

or

\[ |H(j\omega')|^2 = \frac{1}{1 + (\omega')^{2N}} \]

(normalized prototype, \(\omega'_p = 1\))

Maximally flat response
Poles of the Lowpass Prototype (Normalized $\omega_p = 1$)

Butterworth Filter

Magnitude Square Function:

$$\left| H(j\omega') \right|^2 = \frac{1}{1 + \omega'^{2N}}$$

$$H(s)H(-s) = \frac{1}{1 + \omega'^{2N}} \bigg|_{\omega_p=1}$$

$$H(s)H(-s) = \frac{1}{1 + (-s^2)^N}$$
Poles: Solve: $1 + (-s^2)^N = 0$

a. N is odd

$1 - s^{2N} = 0$

$s^{2N} = 1$

Roots:

\[ s_k = 1 \cdot e^{\frac{j(2k\pi)}{2N}} = 1 \cdot e^{\frac{j(k\pi)}{N}} \]

\[ k = 0, 1, 2, \ldots, 2N - 1 \]

b. N is even:

$1 + s^{2N} = 0 \quad s^{2N} = -1$

Roots:

\[ s_k = 1 \cdot e^{\frac{j(\pi + k2\pi)}{2N}} \]

\[ k = 0, 1, 2, \ldots, 2N - 1 \]
Poles of a normalized ($\omega_p' = 1$) Butterworth (LPF) filter lie uniformly spaced on a unit circle in the s-plane at the following locations:

$$s_k = e^{j\pi(2k + N - 1)/2N} = \cos\left[\frac{(2k + N - 1)\pi}{2N}\right] + j\sin\left[\frac{(2k + N - 1)\pi}{2N}\right]$$

$$k = 1, 2, \ldots, N$$

The poles occur in complex conjugate pairs and lie on the left-hand side of the s-plane.

<table>
<thead>
<tr>
<th>TABLE 10.1 BUTTERWORTH PROTOTYPE COEFFICIENTS</th>
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<tbody>
<tr>
<td>$N$</td>
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$H_{LP}(s) = \frac{1}{1 + a_1s + a_2s^2 + \cdots + a_Ns^N}$

Example: $N = 3$

$s_0 = 1, s_1 = 1.e^{j\pi/3}, s_2 = 1.e^{j2\pi/3}$

$s_3 = 1.e^{j\pi}, s_4 = 1.e^{j4\pi/3}, s_5 = 1.e^{j5\pi/3}$

$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$

Butterworth
Filter order $N$
Analog Chebyshev Filters

Define:

\[ |H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 C_N^2(\omega' / \omega_p')} \]

\[ |H_{LP_p}(j\omega')|^2 = \frac{1}{1 + \varepsilon^2 C_N^2(\omega')} \]

(normalized prototype, \( \omega_p' = 1 \))

Where:

\( C_N(\omega') \) is the \( N^{th} \) order chebyshev polynomial.

\[ C_N(\omega) = \cos(N\cos^{-1}\omega') \quad 0 \leq \omega' \leq 1 \]

\[ C_N(\omega) = \cosh(N\cosh^{-1}\omega') \quad \omega' > 1 \]

\( \varepsilon \) is the ripple parameter \( 0 < \varepsilon < 1 \) sets the ripple amplitude in the ripple passband \( 0 \leq \varepsilon \leq 1 \)
The ripple amplitude in dB is given by:

\[
\gamma_{dB} = -10 \log_{10} \left( \frac{1}{1 + \varepsilon^2} \right)
\]

\[
= 10 \log_{10} (1 + \varepsilon^2)
\]

Fig: Chebyshev prototype low pass response
Also,

\[ C_{N+1}(\omega') = 2\omega' C_N(\omega') - C_{N-1}(\omega') \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( C_N(\omega) )</th>
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<tr>
<td>1</td>
<td>( \omega )</td>
</tr>
<tr>
<td>2</td>
<td>( 2\omega^2 - 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( 4\omega^3 - 3\omega )</td>
</tr>
<tr>
<td>4</td>
<td>( 8\omega^4 - 8\omega^2 + 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( 16\omega^5 - 20\omega^3 + 5\omega )</td>
</tr>
<tr>
<td>6</td>
<td>( 32\omega^6 - 48\omega^4 + 18\omega^2 - 1 )</td>
</tr>
<tr>
<td>7</td>
<td>( 64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega )</td>
</tr>
<tr>
<td>8</td>
<td>( 128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1 )</td>
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Chebyshev Filter order \( N \)

\[
N \geq \frac{\cosh^{-1} \left( \frac{A_s}{10^{10}} - 1 \right)}{2 \cosh^{-1} \left( \frac{\omega'_s}{\omega'_p} \right)}
\]
Poles of a normalized ($\omega' = 1$) Chebyshev LPF

$$s_k = \sinh(\alpha) \cos(\beta_k) + j \cosh(\alpha) \sin(\beta_k),$$

where

$$\alpha = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon}\right);$$

$$\beta_k = \frac{(2k + N - 1)\pi}{2N}, \ k = 1, 2, \ldots, N$$

Fig: Prototype Chebyshev denominator polynomials
Mapping analog filters to digital domain:

- Mapping differentials:

Backward: \[ \frac{dy_a(t)}{dt} \bigg|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n] - y[n-1]}{T} \]

Forward: \[ \frac{dy_a(t)}{dt} \bigg|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n+1] - y[n]}{T} \]

\( \nabla^{(1)} \) Backward or forward difference operator

Also, \( \nabla^{(k)}[y[n]] = \nabla^{(1)}[\nabla^{(k-1)}[y[n]]] \)

with \( \nabla^{(0)}(.) = y[n] \)
Analog Filters

\[ H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k} \]

Digital Filters

\[ H(z) = \frac{\sum_{k=0}^{M} d_k \left[ \frac{1 - z^{-1}}{T} \right]^k}{\sum_{k=0}^{N} c_k \left[ \frac{1 - z^{-1}}{T} \right]^k} \]

Mapping Functions

\[ s = \frac{1 - z^{-1}}{T} \]

\[ s = \frac{z - 1}{T} \]

Backward Difference Mapping

Forward Difference Mapping

- Both the mapping functions may produce satisfactory results only for extremely small sampling times and are generally not recommended.
Impulse Invariance Mapping:

\[ H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \quad \leftrightarrow \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \]

- Evaluation – Pole Mapping:

(a) The impulse invariant mapping maps poles from the \( s \)-plane’s \( j \omega' \) axis to the \( z \)-plane’s unit circle.

(b) All \( s \)-plane poles with negative real parts map to \( z \)-plane poles inside the unit circle. In other words, stable analog poles are mapped to stable digital filter poles.

(c) All poles on the right half of the \( s \)-plane, i.e. with positive real parts, map to digital poles outside the unit circle.

The impulse invariant mapping, thus, preserves the stability of the filter.
• Deficiency of the impulse-invariance mapping:

If s-plane poles have the same real parts and imaginary parts that differ by some integer multiples of $2\pi/T$, then there are an infinite number of s-plane poles that map to the same location in the z-plane. This will result in aliased poles in the z-plane.

• Aliasing in digital frequency response:

$$H(e^{j\omega'T}) = \frac{1}{T} H_a(j\omega') + \frac{1}{T} H_a(j\omega' \pm j \frac{2\pi}{T}) + ....$$

Digital frequency response reproduces the analog frequency response every $2\pi/T$.

To avoid aliasing for lower sampling rates, the pole mapping is restricted to the primary strip $-\frac{\pi}{T} \leq \omega' \leq \frac{\pi}{T}$

Thus each horizontal strip of the s-plane of width $2\pi/T$ is mapped onto the entire z-plane.
To prevent significant aliasing:

$$H_a(j\omega') \approx 0 \quad \text{for} \quad |\omega'| > \frac{\pi}{T}$$

Band limit the analog filter.
Example of the impulse-invariance mapping:

Third order Butterworth low-pass filter’s transfer function:

\[ H(s) = \frac{1}{(s + 1)(s^2 + s + 1)} \]

The partial fraction expansion is:

\[ H(s) = \frac{1}{(s + 1)(s + 0.5 - j0.866)(s + 0.5 + j0.866)} \]

\[ = \frac{C_1}{(s + 1)} + \frac{C_2}{(s + 0.5 - j0.866)} + \frac{C_3}{(s + 0.5 + j0.866)} \]

Algebra gives the coefficients as:

\[ = \frac{1}{(s + 1)} + \frac{0.577e^{-j2.62}}{(s + 0.5 - j0.866)} + \frac{0.577e^{j2.62}}{(s + 0.5 + j0.866)} \]

The three poles are:

\[ p_1 = -1, \ p_2 = -0.5 + j0.866, \ p_3 = -0.5 - j0.866 \]
Therefore, the transfer function can be written as:

\[
H(z) = \frac{1}{(1 - e^{-T} z^{-1})} + \frac{1}{(1 - e^{-0.5+j0.866T} z^{-1})} + \frac{1}{(1 - e^{-0.5-j0.866T} z^{-1})}
\]

Simplifying, we get

\[
H(z) = \frac{z}{(z - e^{-T})} + \frac{-z^2 - 1.154e^{-0.5T} \cos(5\pi / 6 + 0.866T)}{(1 - e^{-0.5+j0.866T} z^{-1})}
\]

This can be further simplified and expressed as:

\[
H(z) = \frac{b_0 z^2 + b_1 z}{z^3 + a_1 z^2 + a_2 z + a_3}
\]

where

\[
b_0 = -2e^{-0.5T} \cos(0.866T) + e^{-T} + 1.154e^{-0.5T} \cos(5\pi / 6 + 0.866T)
\]

\[
b_1 = e^{-T} + 1.154e^{-1.5T} \cos(5\pi / 6 + 0.866T)
\]

\[
a_1 = -(e^{-T} + 2e^{-0.5T} \cos(0.866T))
\]

\[
a_2 = e^{-T} + 2e^{-0.5T} \cos(0.866T)
\]

\[
a_3 = -e^{-2T}.
\]
**Bilinear Mapping:**

\[ H(s) = \frac{1}{s} \quad \xrightarrow{\text{Bilinear Mapping}} \quad H(z) = \frac{T}{2} \frac{z + 1}{z - 1} \]

\[ s^{-1} = \frac{T}{2} \frac{z + 1}{z - 1} \quad \text{or} \quad s = \frac{2}{T} \frac{z - 1}{z + 1} \]

- **Evaluation – Pole Mapping:**
  Stable analog filters are mapped from s-plane to z-plane as stable digital filters. Aliasing problem is eliminated.

- **Analog and Digital frequencies relationship for prewarping:**

  \[ \omega' = k \tan(\omega T / 2) \]

  \( \omega' \rightarrow \text{Analog frequency} \)

  \( \omega \rightarrow \text{Digital frequency} \)
Summary of Bilinear Transformation method

For standard, frequency-selective IIR filters, the steps for using the BZT may be summarized as follows:

1. Use the digital filter specifications to find a suitable normalized, prototype, analog lowpass filter, \( H(s) \).

2. Determine and prewarp the bandedge or critical frequencies of the desired filter. For lowpass or highpass filters there is just one bandedge or cutoff frequency (\( \omega_p \), say). For bandpass and bandstop filters, we have the lower and upper passband edge frequencies, \( \omega_{p1} \) and \( \omega_{p2} \), each of which needs to be prewarped (the stopband edge frequencies may also be specified):

\[
\omega_p' = \tan\left(\frac{\omega_p T}{2}\right)
\]

\[
\omega_{p1}' = \tan\left(\frac{\omega_{p1} T}{2}\right); \quad \omega_{p2}' = \tan\left(\frac{\omega_{p2} T}{2}\right)
\]

3. Denormalize the analog prototype filter by replacing \( s \) in the transfer function, \( H(s) \), using one of the following transformations, depending on the type of filter required:

\[
s = \frac{s}{\omega_p'} \quad \text{lowpass to lowpass} \tag{8.21a}
\]

\[
s = \frac{\omega_p'}{s} \quad \text{lowpass to highpass} \tag{8.21b}
\]

\[
s = \frac{s^2 + \omega_0^2}{W_x} \quad \text{lowpass to bandpass} \tag{8.21c}
\]

\[
s = \frac{W_s}{s^2 + \omega_0^2} \quad \text{lowpass to bandstop} \tag{8.21d}
\]

where

\[
\omega_0^2 = \omega_{p2}' \omega_{p1}', \quad W = \omega_{p2}' - \omega_{p1}'
\]

4. Apply the BZT to obtain the desired digital filter transfer function, \( H(z) \), by replacing \( s \) in the frequency-scaled (i.e. denormalized) transfer function, \( H'(s) \), as follows:

\[
s = \frac{z - 1}{z + 1}
\]
Example 1:

Low pass filter: Obtain the transfer function $H(z)$ of the digital low pass filter to approximate the following transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Use Bilinear Transformation method and assume a 3 dB cut off frequency of 150Hz and a sampling frequency of 1.28kHz.

**Soln.** The critical frequency, $\omega_p = 2\pi \times 150 \text{ rad/s}$, and $F_s = 1/T = 1.28\text{kHz}$, giving a prewarped critical frequency of $\omega'_p = \tan(\omega_p T/2) = 0.3857$

The frequency-scaled analog filter is given by

$$H'(s) = H(s)|_{s = \omega_p} = \frac{\omega'_p^2}{(s/\omega_p)^2 + \sqrt{2}s/\omega_p + 1}$$

$$= \frac{\omega'_p^2}{s^2 + \sqrt{2}\omega_p' s + \omega_p'^2} = \frac{0.1488}{s^2 + 0.5455s + 0.1488}$$

Applying the BZT gives

$$H(z) = H'(s)|_{z = 1} = \frac{0.0878z^2 + 0.1756z + 0.0878}{z^2 - 1.0048z + 0.3561}$$

$$\approx \frac{0.0878(z - 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$
Use of BZT and Classical Analog Filters

Analog Filters Review

• Low pass Butterworth filter of order N

\[
|H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_p})^{2N}}
\]  \hspace{1cm} (8.24)

Frequency response

Poles of a normalized \((\omega'_p = 1)\) Butterworth LPF

\[
s_k = e^{j\pi(2k+N-1)/2N} = \cos\left[\frac{(2k + N - 1)\pi}{2N}\right] + j\sin\left[\frac{(2k + N - 1)\pi}{2N}\right]
\]  \hspace{1cm} (8.25)

\begin{align*}
&k = 1, 2, ..., N
\end{align*}

Filter order N

\[
N \geq \frac{\log\left(10^{10} - 1\right)}{\log\left(\frac{\omega'_s}{\omega'_p}\right)}
\]  \hspace{1cm} (8.26)
2. Low pass Chebyshev Type 1 Filter of order N

Frequency response

\[ |H(j\omega')|^2 = \frac{K}{1 + \epsilon^2 C_N^2 (\omega' / \omega_p')^N} \]

(8.27)

Poles of a normalized \((\omega_p' = 1)\) Chebyshev LPF

\[ s_k = \sinh(\alpha) \cos(\beta_k) + j \cosh(\alpha) \sin(\beta_k), \]

where \( \alpha = \frac{1}{N} \sinh^{-1}\left( \frac{1}{\epsilon} \right); \)

\( \beta_k = \frac{(2k + N - 1)\pi}{2N}, k = 1, 2, \ldots, N \)

(8.29)

Filter order N

\[ N \geq \frac{\cosh^{-1}\left( \frac{A_z}{10^{10} - 1} \right)}{2 \cosh^{-1}\left( \frac{\omega_s'}{\omega_p'} \right)} \]

(8.28)
Analog Filters Review contd

3. Low pass Elliptic Filter of order N

Frequency response

\[ |H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 G_N^2(\omega')} \]

(8.30)

Fig: Sketches of frequency response of some classical analog filters
(a) Butterworth response (b) Chebyshev type I © Chebyshev type II
(d) Elliptic
Analog Frequency Transformations

1. Desired Low pass → Low pass prototype

The low pass to low pass transformation is:

\[ s = \frac{s}{\omega_p'} \]  \hspace{1cm} \text{From 8.21 (a)}

Denote frequencies for

LPF \rightarrow \omega_{lp} \text{ and } \text{Prototype} \rightarrow \omega^p

\[ j \omega^p = j \frac{\omega_{lp}}{\omega_p'} \text{ i.e. } \omega^p = \frac{\omega_{lp}}{\omega_p'} \] \hspace{1cm} (8.31)
2. *Desired High pass* $\rightarrow$ *Low pass prototype*

The low pass to high pass transformation is:

\[ s = \frac{\omega_p'}{s} \]

From 8.21 (b)

Denote frequencies for

HPF $\rightarrow$ $\omega_{hp}$ and Prototype $\rightarrow$ $\omega^p$

\[ j\omega^p = -j \frac{\omega_p'}{\omega_{hp}} \quad \text{i.e.} \quad \omega^p = -\frac{\omega_p'}{\omega_{hp}} \quad (8.32) \]
The BZT methodology using classical analog filters

In cases where the prototype lowpass filter does not exist, the stages of the BZT method are:

1. Prewarp the bandedge or critical frequencies of the digital filter as described previously.
2. Find a suitable lowpass prototype analog filter, based on the digital filter specifications and one of the classical filter characteristics. This involves using one of the frequency transformation equations (depending on the type of digital filter – LP, HP, BP or BS) in reverse to determine the specifications of the prototype LP filter. From this the order of the prototype filter and hence its transfer function, $H(s)$, are found.
3. Denormalize the prototype analog LP filter, $H(s)$, by frequency transformation and scaling to obtain a new transfer function, $H'(s)$, as described previously.
4. Apply the BZT to obtain the desired digital filter transfer function, $H(z)$, by replacing $s$ in the frequency scaled transfer function, $H'(s)$, as described previously.
Design Example

Lowpass filter A lowpass digital filter meeting the following specifications is required:

- passband: 0–500 Hz
- stopband: 2–4 kHz
- passband ripple: 3 dB
- stopband attenuation: 20 dB
- sampling frequency: 8 kHz

Determine the following:

1. pass- and stopband edge frequencies for a suitable analog prototype lowpass filter;
2. order, $N$, of the prototype lowpass filter;
3. coefficients, and hence the transfer function, of the discrete-time filter using the bilinear $z$-transform.

Assume a Butterworth characteristic for the filter.

Solution

1. From the specifications, the prewarped frequencies are

$$
\omega'_p = \tan \left( \frac{2\pi \times 500}{2 \times 8000} \right) = 0.198912
$$

$$
\omega'_s = \tan \left( \frac{2\pi \times 2000}{2 \times 8000} \right) = 1
$$

$$
\omega''_s = \frac{\omega'_s}{\omega'_p} = \frac{1}{0.198912} = 5.0273
$$

Thus, the prewarped pass- and stopband edge frequencies for the prototype LP filter are: 0, 1, 5.0273.

2. Next, we use Equation 8.25 and the values of the parameters above to determine the order of the filter.

Now

$$
10^{\omega'_s / 10} - 1 = 10^{5.0273 / 10} - 1 = 99; \quad 10^{\Delta / 10} - 1 = 10^{3 / 10} - 1 = 0.9952623
$$

$$
\log \left( \frac{99}{0.9952623} \right) = 1.997697
$$

For the prototype LPF

$$
\omega''_p = 1; \quad \omega''_s = 5.0273; \quad \log \left( \frac{\omega''_s}{\omega''_p} \right) = 2 \log (5.0273) = 1.40266.
$$

$$
N \approx \frac{1.997697}{1.40266} = 1.424. \text{ Let } N = 2
$$
(3) The poles of the prototype filter are (from Equation 8.26)

\[ s_{p,1} = \cos \left( \frac{(2 + 2 - 1) \pi}{4} \right) + j \sin \left( \frac{(2 + 2 - 1) \pi}{4} \right) = -\frac{\sqrt{2}}{2} + j \frac{\sqrt{2}}{2} \]

\[ s_{p,2} = -\frac{\sqrt{2}}{2} - j \frac{\sqrt{2}}{2} \]

The s-plane transfer function, \( H(s) \), is

\[ H(s) = \frac{1}{(s - s_{p,1})(s - s_{p,2})} = \frac{1}{s^2 + \sqrt{2} s + 1} \]

The frequency scaled s-plane transfer function is

\[ H'(s) = H(s) \left| _{s = \omega_c s_0} \right. = \frac{1}{\left( \frac{s}{\omega_c s_0} \right)^2 + \sqrt{2} \frac{s}{\omega_c s_0} + 1} \]

\[ = \frac{\omega_c^2 s_0^2}{s^2 + \sqrt{2} s \omega_c s_0 + \omega_c^2 s_0^2} \]

Applying the BZT:

\[ H(z) = H'(s) \left| _{s = \frac{z - 1}{z + 1}} \right. = \frac{\omega_c^2}{(\frac{z - 1}{z + 1})^2 + \sqrt{2} \omega_c (\frac{z - 1}{z + 1}) + \omega_c^2} \]

\[ = \frac{\omega_c^2(z + 1)^2}{(z - 1)^2 + \sqrt{2} \omega_c^2(z - 1)(z + 1) + \omega_c^2(z + 1)^2} \]

After simplification and dividing top and bottom by \( z^2 \), we have

\[ H(z) = \frac{\omega_c^2}{1 + \sqrt{2} \omega_c + \omega_c^2} \times \frac{1 + 2 z^{-1} + z^{-2}}{1 + \frac{2(\omega_c^2 - 1)z^{-1}}{1 + \sqrt{2} \omega_c + \omega_c^2} + \frac{(1 - \sqrt{2} \omega_c + \omega_c^2)z^{-2}}{1 + \sqrt{2} \omega_c + \omega_c^2}} \]

Using the values of the parameters

\[ 1 + \sqrt{2} \omega_c + \omega_c^2 = 1.320 87; \quad \omega_c^2 - 1 = -0.960 43 \]

\[ 1 - \sqrt{2} \omega_c + \omega_c^2 = 0.758 285 8; \quad \omega_c^2 = 0.039 565 9 \]

and substituting in the equation above and simplifying, we have

\[ H(z) = \frac{0.029 95(1 + 2 z^{-1} + z^{-2})}{1 - 1.4542 z^{-1} + 0.574 08 z^{-2}} \]
References

   Second Edition