Transform Domain Representation of Discrete Time Signals

The Discrete Fourier Transform

(II)

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Review of DFT (N-Point Transform)

DFT

\[
X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}nk\right)} \quad (1)
\]

IDFT

\[
x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\left(\frac{2\pi}{N}nk\right)} \quad (2)
\]

let \( W_N = e^{-j\left(\frac{2\pi}{N}\right)} \)

\( W_N^{-1} = e^{j\left(\frac{2\pi}{N}\right)} \) ...Then

DFT

\[
X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad (3)
\]

IDFT

\[
x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad (4)
\]
We Have proved:

\[
\sum_{k=0}^{N-1} W_N^k(n-p) = \begin{cases} 
N..p=n \\
0..otherwise
\end{cases}
\]

and they are periodic in \(k\) and in \(n\).

- DFT inherits the periodicity from the periodicity of complex digital exponentials:

From Equation (1)

\[
X(k + N) = \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi}{N}\right)n(k+N)}
\]

\[
= \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi}{N}\right)nk - j\left(\frac{2\pi}{N}\right)nN}
\]
From Equation (2)

\[ x[n + N] = \frac{1}{N} \sum_{n=0}^{N-1} X(k)e^{j\left(\frac{2\pi}{N}kn\right)} \]

Periodicity

\[ X(k+N) = X(k) \quad k = 0, 1, \ldots, N-1 \]
\[ x[n+N] = x[n] \quad n = 0, 1, \ldots, N-1 \]
DFT Properties:

1) DFT is Linear:
\[ \text{DFT}_N \left( a x_1[n] + b x_2[n] \right) \]
\[ = \text{DFT}_N \left( a x_1[n] \right) + \text{DFT}_N \left( b x_2[n] \right) \]
\[ = a X_1(k) + b X_2(k) \]

2) IDFT is Linear:
\[ \text{IDFT}_n \left( a X_1(k) + b X_2(k) \right) \]
\[ = \text{IDFT}_n \left( a X_1(k) \right) + \text{IDFT}_n \left( b X_2(k) \right) \]
\[ = a x_1[n] + b x_2[n] \]

3) DFT is periodic
\[ X(k+N) = X(k) \]

4) IDFT is periodic
\[ x[n] = x[n+N] \]
DFT representations:

The period: \(X(0)\ldots X(N-1)\) standard frequencies span from:

\[0 - 2\pi(N-1)/N\ \text{rad/sample}\]

The period centered around 0:

\[\{X(-N/2)\ldots ,X(0),\ldots \ldots X(N/2-1)\}\]

\{Normalized frequencies - \(\pi\) to \(\pi - 2\pi/N\) rad/sample\}

Also Note:

\[X_{-N/2} = X_{N/2}\]

\[X_{-N/2+1} = X_{N/2+1}\]

\[\ldots \ldots\]

\[X_{-1} = X_{N-1}\]
5) DFT: Time Shift

\[
DFT[x[n - n_0]] = \sum_{n=0}^{N-1} x[n - n_0] e^{-j\left(\frac{2\pi}{N}\right)nk}
\]

\[
= \sum_{<N>} x[m] e^{-j\left(\frac{2\pi}{N}\right)kn} e^{-j\left(\frac{2\pi}{N}\right)kn_0}
\]

\[
= e^{-j\left(\frac{2\pi}{N}\right)kn_0} DFT(x[m])
\]

\[
= e^{-j\left(\frac{2\pi}{N}\right)kn_0} X(k)
\]

DFT: If \( X_2(k) = X_1(k) e^{-j\left(\frac{2\pi}{N}\right)km} \)

Then:

\( x_2[n] = x_1[n \oplus m] \leftarrow \text{circular shift} \)
Circular shift for $m=1$

\[ x_2[n] = x_1[n \oplus m] \leftarrow \text{circular shift} \]
Example:

Let:

If \( x_2[n] = x_1[n+1] \)

Which Implies: \( X_2(k) = X_1(k)e^{j\left(\frac{2\pi}{3}\right)k} \)

6) Evaluation of IDFT from DFT(\( x[n] \))

\[
x[n] = \frac{1}{N} \left[ \sum_{k=0}^{N-1} X^*(k)e^{-j\left(\frac{2\pi}{N}\right)nk} \right]^* \\
= \frac{1}{N} \left[ DFT[X^*(k)] \right]^*
\]
Thus we see that IDFT is the complex conjugate of the DFT of $X^*(k)$ multiplied by $1/N$

7) DFT: Symmetry Properties

$x[n]$ is real

(1) $\text{Re}(X(k)) = \text{Re}(X(N-k))$ \quad k = 1, 2, \ldots N/2-1 \quad N \text{ even}

(2) $\text{Im}(X(k)) = \text{Im}(X(N-k))$ \quad k = 1, 2, \ldots N-1/2 \quad N \text{ odd}

(3) $|X(k)| = |X(N-k)|$ \quad k = 1, 2, \ldots N/2-1 \quad N \text{ even}

(4) $<X(k) = -<X(N-k)$ \quad k = 1, 2, \ldots N-1/2 \quad N \text{ odd}
Example:

\[ X(k) = \sum_{n=0}^{4} x[n] e^{-j\left(\frac{2\pi}{5}\right)kn} \]

Verify:

\[ X(0) = 10 \]

\[ X(1) = 1 + 1e^{-j\left(\frac{2\pi}{5}\right)} + 2e^{-j\left(\frac{4\pi}{5}\right)} + 3e^{-j\left(\frac{6\pi}{5}\right)} + 3e^{-j\left(\frac{8\pi}{5}\right)} = 3.08e^{j0.7\pi} \]

\[ X(2) = 1 + 1e^{-j\left(\frac{4\pi}{5}\right)} + 2e^{-j\left(\frac{8\pi}{5}\right)} + 3e^{-j\left(\frac{12\pi}{5}\right)} + 3e^{-j\left(\frac{16\pi}{5}\right)} = 0.73e^{j0.9\pi} \]

Follows:

\[ X(3) = X^*(2) = 0.73e^{-j0.9\pi} \]

\[ X(4) = X^*(1) = 3.08e^{-j0.7\pi} \]
8) Even functions: If \( x(n) \) is an even function \( x_e(n) \), i.e. \( x_e(n) = x_e(-n) \), then
\[
F_D[x_e(n)] = X_e(k) = \sum_{n=0}^{N-1} x_e(n) \cos(k\Omega nT)
\]

9) Odd functions: If \( x(n) \) is an odd function \( x_o(n) \), i.e. \( x_o(n) = -x_o(-n) \), then
\[
F_D[x_o(n)] = X_o(k) = -j \sum_{n=0}^{N-1} x_o(n) \sin(k\Omega nT)
\]

10) Parseval’s theorem: The normalized energy in the signal is given by either of the expressions
\[
\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2
\]

11) Delta function: \( F_D[\delta(nT)] = 1 \)
12) The linear and circular cross-correlations of two data sequences may be computed using DFTs.

For example, the circular correlation of two finite length periodic sequences $x_{1p}(n)$, $x_{2p}(n)$ can be calculated as

\[
r_{cx_{1p}x_{2p}}(j) = \frac{1}{N} \sum_{n=0}^{N-1} x_{1p}(n)x_{2p}(n+j), \quad j = 0, \ldots, N-1
\]

\[= F_D^{-1}[X_1^*(k)X_2(k)]\]

13) DFTs may also be used in the computation of circular and linear convolutions, for example

\[
x_{3p}(n) = x_{1p}(n) \otimes x_{2p}(n)
\]

\[= F_D^{-1}[X_1(k)X_2(k)]\]

where $\otimes$ denotes circular convolution, and $x_{1p}(n)$, $x_{2p}(n)$, and $x_{3p}(n)$ are finite periodic sequences of equal length.